Duality for normed spaces

4.1. Recall from the lectures that if $1 < p, q < +\infty$ and $1/p + 1/q = 1$, then there exists an isometric isomorphism $\ell^q \xrightarrow{\sim} (\ell^p)^*$. By using a similar argument, construct isometric isomorphisms **(a)** $\ell^\infty \xrightarrow{\sim} (\ell^1)^*$; **(b)** $\ell^1 \xrightarrow{\sim} (c_0)^*$. Does this approach give an isometric isomorphism $\ell^1 \cong (\ell^\infty)^*$?

4.2. Describe explicitly the duals of the following operators:
(a) the diagonal operator on ℓ^p (where $1 \leqslant p < \infty$) or on c_0;
(b) the right shift operator on ℓ^p (where $1 \leqslant p < \infty$) or on c_0;
(c) the operator of "taking the primitive" on $L^2[0, 1]$ (see Exercise 2.6);
(d) the Hilbert-Schmidt integral operator on $L^2(X, \mu)$ (see Exercise 2.8).

4.3-B. Prove that c_0 is not isomorphic to the dual of a normed space.

4.4. Let X be a normed space.
(a) Prove that if X^* is separable, then so is X.
(b) Is the converse true?
(c) Prove that there is no topological isomorphism between $(\ell^\infty)^*$ and ℓ^1.

4.5. Let X be a normed space, and let $i_X : X \to X^{**}$ be the canonical embedding. Prove that for each operator $T \in \mathscr{B}(X, Y)$ the following diagram commutes.

$$\begin{array}{ccc} X^{**} & \xrightarrow{T^{**}} & Y^{**} \\ {\scriptstyle i_X}\uparrow & & \uparrow{\scriptstyle i_Y} \\ X & \xrightarrow{T} & Y \end{array}$$

4.6. Prove that the composition of the canonical embedding $c_0 \to (c_0)^{**}$ and the standard isomorphism $(c_0)^{**} \cong \ell^\infty$ is the inclusion of c_0 into ℓ^∞. Deduce that c_0 is not reflexive.

4.7. Prove that **(a)** a Hilbert space is reflexive; **(b)** ℓ^1 is not reflexive; **(c)** $L^1[a, b]$ is not reflexive; **(d)** $C[a, b]$ is not reflexive.

4.8. Let X be a normed space, and let $i_X : X \to X^{**}$ be the canonical embedding. Find a relation between the operators $i_{X^*} : X^* \to X^{***}$ and $i_X^* : X^{***} \to X^*$.

4.9. **(a)** Prove that a Banach space X is reflexive $\iff$ X^* is reflexive.
(b) Deduce that ℓ^1, ℓ^∞, $L^\infty[a, b]$ are not reflexive.

4.10. Let X and Y be Banach spaces, and let $S \in \mathscr{B}(Y^*, X^*)$. Do we always have $S = T^*$ for some $T \in \mathscr{B}(X, Y)$?

4.11. Identify $(\ell^1)^*$ with ℓ^∞ (see Exercise 4.1), and consider c_0 as a subspace of $(\ell^1)^*$. Find ${}^\perp c_0$ and $({}^\perp c_0)^\perp$.

4.12. Let X be a nonreflexive Banach space. Prove that there exists a closed vector subspace $N \subseteq X^*$ such that $N \neq ({}^\perp N)^\perp$.

4.13. Give an example of an injective operator $T \in \mathscr{B}(X, Y)$ between Banach spaces X and Y such that $\operatorname{Im} T^*$ is not dense in X^*. (*Hint:* X must be nonreflexive, see the lectures.) As a corollary, the equality $\overline{\operatorname{Im}(T^*)} = (\operatorname{Ker} T)^\perp$ can fail in the nonreflexive case.

4.14-B (*weak Johnson's Lemma*)**.** Prove that a sequence $0 \to X \to Y \to Z \to 0$ of Banach spaces is exact if and only if the dual sequence $0 \to Z^* \to Y^* \to X^* \to 0$ is exact.

4.15-B. Let X be a Banach space, and let $X_0 \subseteq X$ be a closed vector subspace. Prove that X is reflexive if and only if X_0 and X/X_0 are reflexive.

Spectra

4.16. Prove that the spectrum of a bijective isometry on a Banach space is contained in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

4.17. Find the point spectrum, the continuous spectrum, and the residual spectrum of the diagonal operator on ℓ^∞.

4.18. Let (X, μ) be a σ-finite measure space, f be an essentially bounded measurable function on X, and M_f be the multiplication operator on $L^p(X, \mu)$ acting by the rule $g \mapsto fg$ (where $1 \leqslant p \leqslant \infty$). Find the point spectrum, the continuous spectrum, and the residual spectrum of M_f.

4.19. Find the point spectrum, the continuous spectrum, and the residual spectrum of the shift operator T_b on $\ell^2(\mathbb{Z})$ acting by the rule $T_b(x)_i = x_{i-1}$ $(i \in \mathbb{Z})$. (*Hint*: replace T_b by a unitary equivalent operator on $L^2(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, or, if you like, on $L^2[-\pi, \pi]$.)

4.20. Find the point spectrum, the continuous spectrum, and the residual spectrum of the left and right shift operators on **(a)** c_0; **(b)** ℓ^1; **(c)-B** ℓ^∞.

4.21. Given $\zeta \in \mathbb{T}$, define the shift operator $T_\zeta \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by $(T_\zeta f)(z) = f(\zeta^{-1} z)$. Find the point spectrum, the continuous spectrum, and the residual spectrum of T_ζ.

4.22 (*the Volterra operator*)**.** Let $I = [a, b]$, let $H = L^2(I)$, and let $K \in L^2(I \times I)$. The *Volterra operator* $V_K \colon L^2(I) \to L^2(I)$ is given by

$$(V_K f)(x) = \int_a^x K(x, y) f(y)\, dy$$

(a) Prove that V_K is quasinilpotent whenever K is bounded.
(b)-B Prove that V_K is quasinilpotent for each $K \in L^2(I \times I)$.