Hilbert spaces

Exercises marked by "**-B**" are optional. If you solve such exercises, you will earn bonus points.

3.1. Show that the norm on the spaces $(\mathbb{C}^n, \|\cdot\|_p)$, ℓ^p, $(C[a,b], \|\cdot\|_p)$, $L^p(X,\mu)$ (where (X,μ) is a measure space containing at least two disjoint measurable sets of positive measure) is not generated by an inner product (unless $n=1$, $p=2$).

3.2. Generalize the parallelogram rule to n vectors.

3.3. Show that the norm on the spaces ℓ^p, $(C[a,b], \|\cdot\|_p)$, $L^p(X,\mu)$ (where (X,μ) is a measure space containing infinitely many disjoint measurable sets of positive measure) is not equivalent to a norm generated by an inner product (unless $p=2$).

3.4. Consider the vector space $H = C[-1,1]$ with the inner product $\langle f \mid g \rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,dt$. Let

$$H_0 = \left\{ f \in H : \int_{-1}^{0} f(t)\,dt = \int_{0}^{1} f(t)\,dt \right\}.$$

(a) Prove that H_0 is a closed vector subspace of H.
(b) Does the equality $H = H_0 \oplus H_0^\perp$ hold?

3.5. Prove that every incomplete inner product space H has a closed vector subspace H_0 such that $H_0 \oplus H_0^\perp \neq H$.

3.6. Let $C_c^\infty(a,b)$ be the space of smooth compactly supported functions on the interval (a,b). Prove that for each $p \in [1,\infty)$ $C_c^\infty(a,b)$ is dense in $L^p[a,b]$.

Definition 3.1. Let $f \in L^2[a,b]$. A function $f' \in L^2[a,b]$ is a *weak derivative* of f if

$$\int_a^b f'\varphi\,dt = -\int_a^b f\varphi' dt$$

for all $\varphi \in C_c^\infty(a,b)$.

3.7. Prove that if $f \in L^2[a,b]$ has a weak derivative f', then f' is unique (as an element of $L^2[a,b]$).

3.8 (*the Sobolev space*)**.** Let $W^{1,2}(a,b)$ denote the space of all $f \in L^2[a,b]$ that have a weak derivative $f' \in L^2[a,b]$. Prove that $W^{1,2}(a,b)$ is a Hilbert space with respect to the inner product

$$\langle f \mid g \rangle = \int_a^b (f\bar{g} + f'\bar{g}')\,dt.$$

3.9 (*the Hardy space*)**.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let H^2 denote the space of holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ satisfying the following condition:

$$\|f\| = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^2\,d\varphi \right)^{1/2} < \infty.$$

Show that the map $f \mapsto (c_n(f))_{n \geqslant 0}$ (where $c_n(f)$ is the nth Taylor coefficient of f at 0) is an isometric isomorphism of $(H^2, \|\cdot\|)$ onto $\ell^2(\mathbb{Z}_{\geqslant 0})$. Hence H^2 is a Hilbert space.

3.10-B (*the Bergman space*)**.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $L_a^2(\mathbb{D})$ denote the space of holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ satisfying the following condition:

$$\|f\| = \left(\int_{\mathbb{D}} |f(x+iy)|^2\,dx\,dy \right)^{1/2} < \infty.$$

Show that $L_a^2(\mathbb{D})$ is a closed vector subspace of $L^2(\mathbb{D})$. Hence $L_a^2(\mathbb{D})$ is a Hilbert space.

The three basic principles

(Hahn-Banach, Banach-Steinhaus, Open Mapping Theorem)

3.11. Let $X = \mathbb{R}^2$ equipped with the norm $\|\cdot\|_p$, and let $X_0 = \{(x,0) : x \in \mathbb{R}\} \subset X$. Define a linear functional $f_0 \colon X_0 \to \mathbb{R}$ by $f_0(x,0) = x$. We clearly have $\|f_0\| = 1$. Describe all "Hahn-Banach extensions" of f_0, i.e., all linear functionals $f \colon X \to \mathbb{R}$ such that $f|_{X_0} = f_0$ and $\|f\| = 1$. (Consider all possible $p \in [1, +\infty]$.)

3.12. Give an example of a normed space X and a pointwise bounded sequence (f_n) in X^* such that (f_n) is not norm bounded.

3.13. Let X, Y, Z be normed spaces.
(a) Prove that a bilinear operator $T \colon X \times Y \to Z$ is continuous if and only if there exists $C \geqslant 0$ such that $\|T(x,y)\| \leqslant C\|x\|\|y\|$ for all $x \in X,\ y \in Y$.
(b) Assume that either X or Y is complete. Prove that each separately continuous bilinear operator $X \times Y \to Z$ is continuous. (The separate continuity means that for each $x_0 \in X$, $y_0 \in Y$ the maps $Y \to Z$, $y \mapsto T(x_0, y)$, and $X \to Z$, $x \mapsto T(x, y_0)$, are continuous.) *Hint:* use the Uniform Boundedness Principle.
(c) Does (b) hold without the completeness assumption?

3.14-B. Let G be a compact topological group, and let π be a representation of G on a Banach space X. Suppose that π is continuous in the sense that the map $G \times X \to X$, $(g,x) \mapsto \pi(g)x$, is continuous. Prove that there exists an equivalent norm $\|\cdot\|_\pi$ on X such that all the operators $\pi(g)$ are isometric with respect to $\|\cdot\|_\pi$. (*Warning:* this has nothing to do with the Haar measure!).

3.15. (a) Deduce the Open Mapping Theorem from the Inverse Mapping Theorem.
(b) Deduce the Inverse Mapping Theorem from the Closed Graph Theorem.
(c)-B Deduce the Uniform Boundedness Principle from the Closed Graph Theorem.

3.16. (a) Give an example of a Banach space X, a normed space Y, and a bijective operator $T \in \mathscr{B}(X,Y)$ such that T^{-1} is unbounded.
(b)-B Give an example of a normed space X, a Banach space Y, and a bijective operator $T \in \mathscr{B}(X,Y)$ such that T^{-1} is unbounded.

3.17. Let $\|\cdot\|$ be a norm on $L^1(\mathbb{R})$ such that $(L^1(\mathbb{R}), \|\cdot\|)$ is complete and such that the convergence $f_n \to f$ with respect to $\|\cdot\|$ implies that $\int_{-\infty}^t f_n(s)\,ds \to \int_{-\infty}^t f(s)\,ds$ for all $t \in \mathbb{R}$. Prove that $\|\cdot\|$ is equivalent to the usual L^1-norm.