

Linear operators

2.1. Let X, Y be normed spaces. Suppose that X is finite-dimensional. Prove that each linear operator $T: X \rightarrow Y$ is bounded.

2.2. Choose $t_0 \in [a, b]$, and consider the linear functional

$$F: (C[a, b], \|\cdot\|_p) \rightarrow \mathbb{K}, \quad F(f) = f(t_0).$$

(a) Find all $p \in [1, +\infty]$ such that F is bounded. (b) Find $\|F\|$.

2.3. Define a linear functional F on $(C[0, 1], \|\cdot\|_\infty)$ by

$$F(f) = 2f(0) - 3f(1) + \int_0^1 f(t) dt.$$

(a) Prove that F is bounded. (b) Find $\|F\|$.

2.4 (*the multiplication operator on $C(I)$*). Let $I = [a, b]$, and let $f \in C(I)$. For each $p \in [1, +\infty]$, define $M_f: C(I) \rightarrow C(I)$ by

$$M_f(g) = fg \quad (g \in C(I)).$$

(a) Prove that M_f is bounded. (b) Find $\|M_f\|$.

Hint: consider separately the cases $p = \infty$ and $p < \infty$.

2.5 (*the multiplication operator on L^p*). Let (X, μ) be a σ -finite measure space, and let $f: X \rightarrow \mathbb{K}$ be an essentially bounded measurable function. For each $p \in [1, +\infty]$, define $M_f: L^p(X, \mu) \rightarrow L^p(X, \mu)$ by

$$M_f(g) = fg \quad (g \in L^p(X, \mu)).$$

(a) Prove that M_f is bounded. (b) Find $\|M_f\|$.

2.6. Let X be either $L^p[0, 1]$ ($1 \leq p < +\infty$) or $C[0, 1]$. Define $T: X \rightarrow X$ by

$$(Tf)(x) = \int_0^x f(t) dt \quad (f \in X).$$

(a) Prove that T is bounded. Find $\|T\|$ in the cases where (b) $X = C[0, 1]$ and (c) $X = L^1[0, 1]$.

Remark. If the above operator T acts on $L^2[0, 1]$, then $\|T\| = 2/\pi$. We will be able to prove this in due course.

2.7 (*the integral operator on $C(I)$*). Let $I = [a, b]$, and let $K \in C(I \times I)$. Define $T: C(I) \rightarrow C(I)$ by

$$(Tf)(x) = \int_a^b K(x, y)f(y) dy.$$

Prove that T takes $C(I)$ to $C(I)$, that T is bounded, and that $\|T\| \leq \|K\|_\infty(b - a)$.

2.8 (*the Hilbert-Schmidt integral operator*). Let (X, μ) be a σ -finite measure space, and let $K \in L^2(X \times X, \mu \times \mu)$. Define $T: L^2(X, \mu) \rightarrow L^2(X, \mu)$ by

$$(Tf)(x) = \int_X K(x, y)f(y) d\mu(y).$$

Prove that the above integral exists for almost all $x \in X$, that T takes $L^2(X, \mu)$ to $L^2(X, \mu)$, that T is bounded, and that $\|T\| \leq \|K\|_2$.

Banach spaces

- 2.9.** Show that c_0 is a closed vector subspace of ℓ^∞ . As a consequence, c_0 is a Banach space.
- 2.10.** Show that $(c_{00}, \|\cdot\|_p)$ is not complete for each $p \in [1, +\infty]$, Show that $(\ell^p, \|\cdot\|_q)$ is not complete if $q > p$. Describe the completions of these spaces.
- 2.11.** Show that the dimension of an infinite-dimensional Banach space is uncountable.
- 2.12.** For each $p < \infty$, construct a Cauchy sequence in $(C[a, b], \|\cdot\|_p)$ which does not converge. Describe the completion of $(C[a, b], \|\cdot\|_p)$.
- 2.13. (a)** Prove that $C^n[a, b]$ is a Banach space with respect to the norm $\|f\| = \max_{0 \leq k \leq n} \|f^{(k)}\|_\infty$.
- (b)** Is $C^n[a, b]$ complete with respect to the sup-norm? Describe the completion of this space.
- 2.14.** Let (X, μ) be a measure space. Prove that $L^\infty(X, \mu)$ is a Banach space.