Introduction to Functional Analysis

Exam 24.12.2019

1. The dual of a normed space. The Riesz Representation Theorem for linear functionals on a Hilbert space.
2. The dual of a normed space. The dual of ℓ^p $(1 < p < \infty)$.
3. The dual of an operator between normed spaces. The norm of the dual operator. Algebraic properties of the operation $T \mapsto T^*$ of taking the dual. The dual of a topological (resp. isometric) isomorphism is an isomorphism of the same type. Similar and isometrically equivalent operators. The duals of diagonal and shift operators on ℓ^p $(1 \leqslant p < \infty)$.
4. The canonical embedding of a normed space into the bidual. The "naturality" of the canonical embedding (a relation between the operators T and T^{**}). Reflexive Banach spaces. Examples.
5. The annihilator of a subset of a normed space. The preannihilator of a subset of the dual space. Relations to orthogonal complements. Basic properties of annihilators and preannihilators. The preannihilator of the annihilator equals the closure of the linear span. A corollary: the density criterion for a vector subspace of a normed space.
6. The annihilator of a subset of a normed space. The preannihilator of a subset of the dual space. The duals of a subspace and of a quotient modulo a closed subspace. Relations between kernels and images of operators and of their duals. A duality between injective operators and operators with dense image.
7. A duality between topologically injective and surjective operators on Banach spaces.
8. The spectrum of an algebra element. The spectra of elements of $\mathbb{C}^X$, $\ell^\infty(X)$, $L^\infty(X, \mu)$. The behavior of the spectrum under homomorphisms. Spectrally invariant subalgebras.
9. The polynomial spectral mapping theorem. The spectrum of the inverse element.
10. Banach algebras. Examples. Properties of the group of invertibles in a Banach algebra. The automatic continuity of characters (i.e., of $\mathbb{C}$-valued homomorphisms). The compactness of the spectrum of a Banach algebra element.
11. The resolvent function and its properties. The nonemptiness of the spectrum of a Banach algebra element. The Gelfand–Mazur theorem.
12. The spectral radius of a Banach algebra element. Examples. The Beurling–Gelfand formula.
13. The point spectrum, the continuous spectrum, and the residual spectrum of a bounded linear operator. An example: calculating the parts of the spectrum for the diagonal operator.
14. The spectrum of the dual operator. Inclusions between the parts of the spectrum of an operator and of the dual operator. An example: the parts of the spectrum of the shift operators on ℓ^p, $1 < p < \infty$.
15. The Riesz lemma on an ε-perpendicular. The noncompactness of the sphere in an infinite-dimensional normed space.

16. Compact operators: definition, basic examples and counterexamples. Properties of the set of compact operators (closed vector subspace of $\mathscr{B}(X,Y)$; the product of a compact operator and a bounded operator is compact). The compactness criterion for the diagonal operator on ℓ^p.

17. Schauder's theorem on the compactness of the dual operator. The approximation of Hilbert-space-valued compact operators by finite rank operators.

18. The adjoint of an operator between Hilbert spaces. A characterization of the adjoint operator in terms of inner products. Basic properties of the operation of taking the adjoint (algebraic properties, $\|T^*\| = \|T\|$, the C^*-property).

19. Selfadjoint operators. The spectrum of a selfadjoint operator is real. Corollaries: the residual spectrum is empty; eigenvectors corresponding to different eigenvalues are orthogonal. The spectral radius of a selfadjoint operator equals the norm. A relation between invariant subspaces on an operator and of its adjoint. The orthogonal complement of an invariant subspace of a selfadjoint operator is invariant.

20. The quadratic form of a selfadjoint operator. The norm of a selfadjoint operator in terms of its quadratic form. The norm of a compact selfadjoint operator in terms of eigenvalues.

21. Properties of eigenspaces and eigenvalues of a compact selfadjoint operator. The Hilbert–Schmidt theorem.