Introduction to Functional Analysis

Midterm exam 01.11.2019

1. Normed spaces. Examples: norms on finite-dimensional spaces, ℓ^p, c_0, $\ell^\infty(X)$, $C_b(X)$, $C^n[a,b]$, $L^p(X,\mu)$.
2. Bounded linear operators. Characterizations of bounded linear operators. Boundedness = continuity. The norm of a bounded linear operator. Examples of bounded linear operators: multiplication operators, shift operators, integral operators.
3. The domination relation and the equivalence of norms on a vector space. Any two norms on a finite-dimensional vector space are equivalent.
4. Banach spaces. Examples: finite-dimensional spaces, $\ell^\infty(X)$, $C_b(X)$, ℓ^p, $L^p(X,\mu)$. A direct proof of the completeness of ℓ^p which does not rely on the completeness of $L^p(X,\mu)$.
5. The space $\mathscr{B}(X,Y)$ of bounded linear operators is complete whenever Y is complete.
6. The "extension by continuity" theorem.
7. Completions of normed spaces: existence, universal property, uniqueness, functoriality.
8. Open operators and coisometries. Characterizations of open operators. Quotients of normed spaces. The universal property of quotients. Corollaries (in particular, conditions under which $X/\operatorname{Ker} T \cong \operatorname{Im} T$ topologically or isometrically). The completeness of quotients.
9. Inner product spaces. The Cauchy-Bunyakowski-Schwarz inequality. The norm generated by an inner product. Hilbert spaces. Examples.
10. Orthogonal complements and their basic properties. Orthogonal projections. A characterization of orthogonal projections in terms of nearest points. The existence of orthogonal projections onto a closed subspace of a Hilbert space. The orthogonal complement theorem.
11. Orthonormal families in inner product spaces. Examples. The cardinality of orthonormal families in a separable space. Fourier coefficients. An explicit formula for the projection onto a finite-dimensional subspace in terms of Fourier coefficients. Bessel's inequality. Fourier series and their elementary properties (uniqueness, Parseval's identity).
12. Orthonormal bases, total orthonormal families, maximal orthonormal families. Relations between these notions (for general inner product spaces and for Hilbert spaces).
13. Orthogonalization. The existence of an orthonormal basis in a separable inner product space. Classification of separable Hilbert spaces. The Riesz-Fischer theorem.
14. The Hanh-Banach theorem for real vector spaces and for linear functionals dominated by sublinear functionals.
15. Relations between complex linear and real linear functionals on a complex vector space. The Hanh-Banach theorem for real or complex seminormed spaces. Corollaries (for normed spaces): extension of bounded linear functionals, a "dual" formula for the norm, linear functionals separate the points, linear functionals separate the points and vector subspaces.

16. Barrels in normed spaces. The barrel lemma for Banach spaces. The Uniform Boundedness Principle (the Banach-Steinhaus theorem).

17. The Open Mapping Theorem, the Inverse Mapping Theorem, the Closed Graph Theorem.