13.1. Let G be a locally compact abelian group, and let $f, g \in L^1(G)$. Show that
(a) $(L_x f)\hat{} = \varepsilon_x \hat{f} \quad (x \in G)$;
(b) $(\chi f)\hat{} = L_{\chi^{-1}} \hat{f} \quad (\chi \in \hat{G})$;
(c) $L_x(f * g) = L_x f * g = f * L_x g \quad (x \in G)$.

13.2. Let G be a locally compact abelian group. Show that for every $f \in L^1(G)$, $g \in L^2(G)$ we have $(f * g)\hat{} = \hat{f}\hat{g}$.

13.3. Let G be a locally compact group. The *reduced group C^*-algebra* of G is the C^*-subalgebra $C^*_r(G)$ of $\mathscr{B}(L^2(G))$ generated by the image of the left regular representation $\lambda \colon L^1(G) \to \mathscr{B}(L^2(G))$. Prove that, if G is abelian, then the canonical map $C^*(G) \to C^*_r(G)$ is an isomorphism.

13.4. Let G be a locally compact abelian group. Show that
(a) G is compact iff $\widehat{G}$ is discrete;
(b) G is discrete iff $\widehat{G}$ is compact.

13.5. Let G be a compact abelian group equipped with the normalized Haar measure (i.e., the measure of G is 1). Show that the Plancherel measure on $\widehat{G}$ is the counting measure.

13.6. Let (G_i) be a family of compact abelian groups. Construct a topological isomorphism $(\prod G_i)\hat{} \cong \bigoplus \widehat{G}_i$ (where $\bigoplus \widehat{G}_i$ is equipped with the discrete topology).

13.7. Construct a (non-topological) isomorphism between $\widehat{\mathbb{Z}}_p$ (see Exercise 4.10) and a certain subgroup of $\mathbb{T}$.

13.8. (a) Given $x = \sum_k a_k p^k \in \mathbb{Q}_p$ (see Exercise 4.10), let $\lambda(x) = \sum_{k<0} a_k p^k \in \mathbb{Q}$. Define a character $\chi_x \colon \mathbb{Q}_p \to \mathbb{T}$ by $\chi_x(y) = e^{2\pi i \lambda(xy)}$. Show that χ_x is continuous, and that the map $\mathbb{Q}_p \to \widehat{\mathbb{Q}}_p$, $x \mapsto \chi_x$, is a topological isomorphism.
(b) Let μ denote the Haar measure on $\mathbb{Q}_p$ normalized in such a way that $\mu(\mathbb{Z}_p) = 1$ (see Exercise 4.11). Identify $\widehat{\mathbb{Q}}_p$ with $\mathbb{Q}_p$, as in (a). Find the respective Plancherel measure on $\mathbb{Q}_p$.

13.9. Let G be a locally compact abelian group. Given a closed subgroup $H \subset G$, define the *annihilator* of H by $H^\perp = \{\chi \in \widehat{G} : \chi|_H = 1\}$.
(a) Show that $H^{\perp\perp} = H$ (here $\widehat{\widehat{G}}$ is canonically identified with G).
(b) Construct topological isomorphisms $\widehat{G/H} \cong H^\perp$, $\widehat{H} \cong \widehat{G}/H^\perp$.
(c) A short exact sequence $1 \to G' \xrightarrow{i} G \xrightarrow{p} G'' \to 1$ of locally compact abelian groups is *strictly exact* if i is a topological embedding and p is a quotient map. Show that a sequence of the above form is strictly exact iff the dual sequence is strictly exact.