9.1. Prove that for each unital algebra A and each $a \in A$ we have $\sigma_{A_+}(a) = \sigma(a) \cup \{0\}$.

9.2. Let $c_{00} \subset c_0$ denote the ideal of finite sequences (i.e., of those sequences $a = (a_n)$ such that $a_n = 0$ for all but finitely many $n \in \mathbb{N}$). Prove that c_{00} is not contained in a maximal ideal of c_0.

9.3. Let $A = \{f \in C[0,1] : f(0) = 0\}$, and let $I = \{f \in A : f$ vanishes on a neighborhood of $0\}$. Prove that I is not contained in a maximal ideal of A.

9.4. Consider the Banach algebra $\ell^2 = \ell^2(\mathbb{N})$ with pointwise multiplication. Show that ℓ^2 has maximal ideals which are not modular.

9.5. A commutative algebra A is *semisimple* if the intersection of all maximal modular ideals of A (the *Jacobson radical* of A) is $\{0\}$. Show that every homomorphism from a Banach algebra to a commutative semisimple Banach algebra is continuous.

9.6. Let A be a commutative algebra, and I be a maximal ideal of A. Prove that I is either modular or a codimension 1 ideal containing $A^2 = \operatorname{span}\{ab : a, b \in A\}$.

9.7. Let A be a commutative algebra, and let $\operatorname{Max}_+(A) = \operatorname{Max}(A) \cup \{A\}$. Prove that the map $\operatorname{Max}(A_+) \to \operatorname{Max}_+(A)$, $I \mapsto I \cap A$, is a bijection.

9.8. Let A be a commutative Banach algebra, and let I be a closed ideal of A.
(a) Construct a homeomorphism between $\operatorname{Max}(A/I)$ and a closed subset of $\operatorname{Max}(A)$.
(b) Show that each nonzero character $I \to \mathbb{C}$ uniquely extends to a character $A \to \mathbb{C}$. Show that the resulting map $\operatorname{Max}(I) \to \operatorname{Max}(A)$ is a homeomorphism onto an open subset of $\operatorname{Max}(A)$.

9.9. Let A be a commutative Banach algebra. Show that the Gelfand transform $\Gamma\colon A \to C_0(\operatorname{Max} A)$ is a topological embedding if and only if there exists $c > 0$ such that $\|a^2\| \geqslant c\|a\|^2$ for all $a \in A$.

9.10. Construct a commutative Banach algebra A such that for each $t \in [0,1]$ there exists a character χ of A with $\|\chi\| = t$.