8.1. A unital commutative algebra A is *local* if A has a unique maximal ideal. Construct a local Banach algebra without zero divisors.

Hint. Consider the subalgebra of $\mathbb{C}[[z]]$ that consists of formal series $a = \sum c_n z^n$ satisfying $\|a\| = \sum |c_n| w_n < \infty$. Here (w_n) is a sequence of positive numbers satisfying some special conditions.

8.2. Let $V\colon L^2[0,1] \to L^2[0,1]$ denote the operator given by

$$(Vf)(x) = \int_0^x f(t)\,dt \qquad (f \in L^2[0,1]).$$

Show that the unital Banach subalgebra of $\mathscr{B}(L^2[0,1])$ generated by V (i.e., the smallest closed subalgebra of $\mathscr{B}(L^2[0,1])$ containing V and the identity operator) is local.

8.3. Let $\mathscr{O}(\mathbb{C})$ be the algebra of holomorphic functions on $\mathbb{C}$ equipped with the norm $\|f\| = \sup_{|z|\leqslant 1} |f(z)|$.

(a) Is $\mathscr{O}(\mathbb{C})$ a Banach algebra?

(b) Show that $\mathscr{O}(\mathbb{C})$ has a dense maximal ideal of infinite codimension.

8.4. (a) Let A be a Banach algebra, $a, b \in A$, $ab = ba$. Prove that $r(a+b) \leqslant r(a) + r(b)$ and $r(ab) \leqslant r(a)r(b)$ (where r is the spectral radius).

(b) Does (a) hold if we drop the assumption that $ab = ba$?

8.5. Let X be a compact Hausdorff topological space. For each closed subset $Y \subset X$ let $I_Y = \{f \in C(X) : f|_Y = 0\}$. Prove that the assignment $Y \mapsto I_Y$ is a 1-1 correspondence between the collection of all closed subsets of X and the collection of all closed ideals of $C(X)$.

8.6. Describe the maximal spectrum and the Gelfand transform for the algebras **(a)** $C^n[0,1]$; **(b)** $\mathscr{A}(\bar{\mathbb{D}})$; **(c)** $\mathscr{P}(\mathbb{T})$ (see Exercise 6.3).

8.7. Let $A(\mathbb{T}) = \{f \in C(\mathbb{T}) : \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty\}$, where $\hat{f}(n)$ is the nth Fourier coefficient of f w.r.t. the trigonometric system (e_n) on $\mathbb{T}$ (i.e., $e_n(z) = z^n$ for all $z \in \mathbb{T}$ and $n \in \mathbb{Z}$). Prove that $A(\mathbb{T})$ is a spectrally invariant subalgebra of $C(\mathbb{T})$.

8.8. Let A and B be commutative unital Banach algebras, and let $\varphi\colon A \to B$ be a continuous unital homomorphism such that $\overline{\varphi(A)} = B$. Show that $\varphi^*\colon \operatorname{Max}(B) \to \operatorname{Max}(A)$ is a topological embedding.