6.1. Let G be a locally compact group. As was shown in the lectures, $L^1(G)$ is a Banach algebra under convolution.
(a) Show that $L^1(G)$ is a Banach $*$-algebra w.r.t. the involution $f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})$ ($f \in L^1(G)$, $x \in G$).
(b) Show that $L^1(G)$ (equipped with the standard L^1-norm and with the involution defined in (a)) is not a C^*-algebra unless $G = \{e\}$.
(c) Show that $L^1(G)$ is commutative if and only if G is commutative.
(d) Show that $L^1(G)$ is unital if and only if G is discrete.

6.2. Let G be a locally compact group, and let $p, q \in (1, +\infty)$ satisfy $1/p + 1/q = 1$. Show that, for each $f \in L^p(G)$ and $g \in L^q(G)$, the convolution $f * Sg$ (where $(Sg)(x) = g(x^{-1})$) is defined everywhere on G, belongs to $C_0(G)$, and that $\|f * Sg\|_\infty \leqslant \|f\|_p \|g\|_q$.

6.3. Let $\mathscr{P}(\mathbb{T})$ denote the closure of $\mathbb{C}[z]$ in $C(\mathbb{T})$, where z is the coordinate on $\mathbb{C}$. Recall that the *disk algebra* $\mathscr{A}(\bar{\mathbb{D}})$ consists of those $f \in C(\bar{\mathbb{D}})$ that are holomorphic on the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Show that
(a) each $f \in \mathscr{P}(\mathbb{T})$ uniquely extends to $\tilde{f} \in \mathscr{A}(\bar{\mathbb{D}})$;
(b) the map $f \mapsto \tilde{f}$ is an isometric isomorphism of $\mathscr{P}(\mathbb{T})$ onto $\mathscr{A}(\bar{\mathbb{D}})$.

6.4. Show that **(a)** $C^n[a, b]$ ($n \geqslant 1$) and **(b)** $\mathscr{A}(\bar{\mathbb{D}})$ are Banach $*$-algebras, but are not C^*-algebras. (Recall that the involution on $C^n[a, b]$ is given by $f^*(t) = \overline{f(t)}$, while the involution on $\mathscr{A}(\bar{\mathbb{D}})$ is given by $f^*(z) = \overline{f(\bar{z})}$.)

6.5. Let A be a normed algebra, and let (e_α) be a bounded approximate identity in A. Show that
(a) if B is a normed algebra and $\varphi \colon A \to B$ is a continuous homomorphism such that $\overline{\varphi(A)} = B$, then $(\varphi(e_\alpha))$ is a bounded approximate identity in B;
(b) if A is a normed $*$-algebra, then $(e_\alpha^* e_\alpha)$ is a bounded approximate identity in A.

6.6. Let X be a locally compact Hausdorff topological space.
(a) Construct a bounded approximate identity in $C_0(X)$.
(b) Show that $C_0(X)$ has a sequential bounded approximate identity if and only if X is σ-compact.

6.7. Let H be a Hilbert space.
(a) Construct a bounded approximate identity in $\mathscr{K}(H)$.
(b) Show that $\mathscr{K}(H)$ has a sequential bounded approximate identity if and only if H is separable.

6.8. Let G be a locally compact group, and let (u_i) be a Dirac net in $L^1(G)$. Identify $L^1(G)$ with a subspace of $C_0(G)^*$ via $f \mapsto I_f$, where $I_f(g) = \int_G fg\, d\mu$ ($g \in C_0(G)$). Show that (u_i) converges to the evaluation functional $g \mapsto g(e)$ (the "Dirac δ-function") w.r.t. the weak* topology on $C_0(G)^*$.