2.1. Prove that the homomorphisms $\mathbb{T} \to \widehat{\mathbb{Z}}$, $\mathbb{Z} \to \widehat{\mathbb{T}}$, and $\mathbb{R} \to \widehat{\mathbb{R}}$ that were constructed at the lecture are indeed topological isomorphisms.

2.2. Prove that the characters of $\mathbb{T}$ form an orthonormal set in $L^2(\mathbb{T})$.

2.3. Prove that the Fourier transform $\mathscr{F}_{\mathbb{T}}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is not surjective. (*Hint:* the dual spaces of $L^1(\mathbb{T})$ and $c_0(\mathbb{Z})$ are not isomorphic.)

For each $k \in \mathbb{Z}$ define $\chi_k\colon \mathbb{T} \to \mathbb{T}$ by $\chi_k(z) = z^{-k}$ (recall that the functions χ_k are precisely the characters of $\mathbb{T}$). The function $D_n = \sum_{k=-n}^{n} \chi_k$ on $\mathbb{T}$ is called the *Dirichlet kernel*. For an arbitrary $f \in L^1(\mathbb{T})$ set $S_n f = \sum_{k=-n}^{n} \hat{f}(k)\chi_{-k}$.

2.4. (a) Prove that for each $f \in L^1(\mathbb{T})$ we have $S_n f = f * D_n$. (*Hint:* compare the Fourier transforms of these functions).

(b) Let $g \in L^1(\mathbb{T})$. Define a functional $\varphi_g\colon C(\mathbb{T}) \to \mathbb{C}$ by the formula $\varphi_g(f) = \int_{\mathbb{T}} f(z)g(z)\, d\mu(z)$. Prove that $\|\varphi_g\| = \|g\|_1$.

(c) Prove that

$$D_n(e^{it}) = 1 + 2\cos t + 2\cos 2t + \cdots + 2\cos nt = \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin \frac{t}{2}}.$$

Deduce that $\|D_n\|_1 \to \infty$ as $n \to \infty$.

(d) Prove that there exists a function $f \in C(\mathbb{T})$ whose Fourier series diverges at $z = 1$. It follows that $\mathscr{F}_{\mathbb{Z}}\colon \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ is not surjective. (*Hint:* It follows from (a) that $(S_n f)(1) = \varphi_{D_n}(f)$. Then use (b), (c) and the Banach-Steinhaus theorem.)

2.5. Compute $\|\hat{D}_n\|_\infty$. Combine this with part (c) of the previous exercise to find another proof of the fact that the Fourier transform $\mathscr{F}_{\mathbb{T}}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is not surjective.

2.6*. Prove that the series

$$\sum_{n=2}^{\infty} \frac{z^n + z^{-n}}{n \ln n}$$

converges uniformly on $\mathbb{T}$ to a continuous function whose Fourier series is not absolutely convergent. This gives yet another proof of the fact that the Fourier transform $\mathscr{F}_{\mathbb{Z}}\colon \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ is not surjective.

2.7. Fix $a \in \mathbb{T}$ and for any $f \in L^2(\mathbb{T})$ and for any $n \in \mathbb{N}$ define

$$f_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} f(a^k z).$$

Prove that the sequence (f_n) converges in $L^2(\mathbb{T})$ to some f. Find f.

2.8. The *convolution* of $f, g \in \ell^1(\mathbb{Z})$ is the function $f * g\colon \mathbb{Z} \to \mathbb{C}$ defined by the formula

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(k)g(n-k) \qquad (n \in \mathbb{Z}). \tag{1}$$

(a) Prove that the series (1) converges absolutely, that $f * g \in \ell^1(\mathbb{Z})$, that $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$, and that $(\ell^1(\mathbb{Z}), *)$ is a commutative unital algebra containing the group algebra $\mathbb{C}\mathbb{Z}$ as a subalgebra.

(b) Prove that for any $f, g \in \ell^1(\mathbb{Z})$ we have $(f * g)\hat{} = \hat{f}\hat{g}$ (i.e., the Fourier transform $\mathscr{F}\colon \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ is an algebra homomorphism).

2.9. The *convolution* of $f, g \in L^1(\mathbb{T})$ is a function $f * g\colon \mathbb{T} \to \mathbb{C}$ defined by the formula

$$(f * g)(z) = \int_{\mathbb{T}} f(\zeta) g(\zeta^{-1} z)\, d\mu(\zeta) \qquad (z \in \mathbb{T}), \tag{2}$$

where μ is the standard measure on $\mathbb{T}$ (arch length/2π).
(a) Prove that the integral (2) exists for almost all $z \in \mathbb{T}$, that $f*g \in L^1(\mathbb{T})$, that $\|f*g\|_1 \leqslant \|f\|_1 \|g\|_1$, and that $(L^1(\mathbb{T}), *)$ is a commutative non-unital algebra.
(b) Prove that for any $f, g \in L^1(\mathbb{T})$ we have $(f * g)\widehat{\ } = \hat{f}\hat{g}$ (i.e., the Fourier transform $\mathscr{F} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is an algebra homomorphism).

2.10 (the *Fourier inversion formula* on $\mathbb{T}$)**.** Let $A(\mathbb{T}) = \mathscr{F}_{\mathbb{Z}}(\ell^1(\mathbb{Z}))$. By the previous exercises, $A(\mathbb{T})$ is a proper dense subalgebra of $C(\mathbb{T})$ (the *Fourier algebra* of $\mathbb{T}$, or the *Wiener algebra*). Show that the following properties of $f \in L^1(\mathbb{T})$ are equivalent:

(i) $\hat{f} \in \ell^1(\mathbb{Z})$.
(ii) There is a (necessarily unique) $f_0 \in A(\mathbb{T})$ such that $f = f_0$ a.e. on $\mathbb{T}$.

Show that, if f satisfies the above conditions, then $f_0(z) = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n$ for all $z \in \mathbb{T}$. In other words, $\mathscr{F}_{\mathbb{T}}\colon A(\mathbb{T}) \to \ell^1(\mathbb{Z})$ is a bijection, and $\mathscr{F}_{\mathbb{Z}} \mathscr{F}_{\mathbb{T}} = S_{\mathbb{T}}$ (where $S_{\mathbb{T}}$ is given by $(S_{\mathbb{T}} f)(z) = f(z^{-1})$).