LECTURE 8.

ABSTRACT. Parking functions.

A line of cars numbered $1, \dots, n$ is trying to park at a lot containing n spots numbered $0, \dots, n-1$, starting from the entrance. Each car i has its own favorite spot $a_i \in \{0, \dots, n-1\}$. Once at the lot, the car proceeds to its favorite lot. If it is free, the car parks there. If not, the car goes further to the lots with bigger numbers (no U-turns are possible), and parks on the first available spot it sees. If the car reaches the end of the lot (the spot $n-1$) and finds no free spots, the parking process fails. The sequence $a_1, \dots, a_n$ is called a parking function if the parking process is successful.

Example 1. $0, \dots, 0$ is a parking function: the first car parks at the spot 0 (its favorite). The second one finds its favorite spot 0 occupied, so it proceeds further and parks at the first available spot, i.e. 1. The third car parks at 2, etc., and the n-th car parks at $n-1$, so the process is successful.

Example 2. Any sequence $a_1, \dots, a_n$ containing two (or more) terms $a_{i_1} = a_{i_2} = n-1$ is not a parking function. Indeed, let $i_1 < i_2$. The car number i_1 goes to the spot number $n-1$. If it is occupied (by some car number $i < i_1$ then the parking process fails: the car i_1 cannot go any further. If $n-1$ is still available then the car i_1 parks on it. Then the car i_2 will find its favorite spot $n-1$ occupied, and the parking process will fail nevertheless.

Theorem 1. *A sequence $a_1, \dots, a_n$ is a parking function if and only if for any $k = 0, \dots, n-1$ the number of $i \in \{1, \dots, n\}$ such that $a_i \geq k$, does not exceed $n-k$.*

Proof. The “only if” part: suppose the condition fails for some k: $\#A_k > n-k$ where $A_k = \{i \mid a_i \geq k\}$, but the parking process is successful. Any car eventually parks at the spot with the number greater or equal to its preferred number; so, in a successful parking the cars with $i \in A_k$ will park at the spots numbered $k+1, \dots, n$. But there are $n-k$ such spots and $\#A_i > n-k$ such cars, so the parking is impossible.

The “if” part: suppose the parking process breaks on the p-th car, $1 \leq p \leq n-1$. The number of cars is equal to the number of spots, so before the p-th car attempts to park, the lot has some free spots; let $k-1$ be the biggest number among them; $k-1 < a_p$, else the p-th car will park successfully. The spots $k, \dots, n-1$ are all occupied, so among the first $p-2$ cars there are at least $n-k$ cars i such that $a_i \geq k$ (a car that fails to park on its favorite spot cannot pass a free spot $k-1$ moving further, so the spots $k, \dots, n-1$ must be occupied by cars with the favorite spots greater or equal to k). Also $a_p \geq k$, and therefore $\#A_k > n-k$ — the condition is not satisfied. □

Corollary 1. *Whether $a_1, \dots, a_n$ is a parking function or not, depends only on the multiset of values $a_1, \dots, a_n$, and not on the order of terms in the sequence. In other words, if one permutes the terms of a parking function in any way, it remains a parking function. (But the number b_i of the spot where the car number i will park, may change!)*

The standard formulation of Corollary 1 is: the group of permutations S_n acts on the set of parking functions permuting their terms. We say that the two parking functions are equivalent (standard term: belong to the same orbit of the action described above) if one can be obtained from the other by permutation of terms. All parking functions are split into several equivalence classes.

Theorem 2. *The number of parking functions $a_1, \dots, a_n$ is equal to $(n+1)^{n-1}$. The number of equivalence classes is equal to the Catalan number.*

Proof. Redesign the parking lot: add one more spot number n, and join it with a lane with the entrance; so, the car passing the occupied spot n will move to the spot number 1, and then further (if necessary). The number of cars waiting to park is still n, but n can now be a favorite spot, too (so $a_i \in \{0, \dots, n\}$ for all $i = 1, \dots, n$).

In the new conditions any sequence $a_1, \dots, a_n$ will give rise to a successful parking, and exactly one spot $F(a_1, \dots, a_n) \in \{0. \dots . n\}$ will remain free (there are totally $n+1$ spots and n cars). Prove two properties of the function F:

Lemma 1. *$a_1, \dots, a_n$ is a parking function (in the original sense) if and only if $F(a_1, \dots, a_n) = n$.*

Proof. If $a_1, \dots, a_n$ is a parking function, the cars will successfully park on the spots $0, \dots, n-1$; since the spot n is after them all, its existence will not change the parking process. Therefore it will remain unoccupied: $F(a_1, \dots, a_n) = n$.

Let now $F(a_1, \dots, a_n) = n$, so the spot n is not occupied. First of all it means that $a_1, \dots, a_n \in \{0, \dots, n-1\}$: a favorite spot of any car will always be occupied. A car that failed to park on its favorite spot cannot pass a free spot n; hence, no car would use the lane joining the spot n with the entrance. In other words, the parking process will be the same as it would have beed before the redesign. Since the parking process is successful, $a_1, \dots, a_n$ is a parking function. $\square$

Lemma 2. *Let $k \in \{0, \dots, n-1\}$, and let $a'_1, \dots, a'_n$ be a sequence where $a'_i = a_i + k \bmod (n+1)$ for all i. Then $F(a'_1, \dots, a'_n) = F(a_1, \dots, a_n) + k \bmod (n+1)$.*

(Here $0, \dots, n$ are interpreted as residues modulo $n+1$.)

Proof. Prove using induction on $m = 1, \dots, n$ that the positions b'_m and b_m where the m-th car will eventually park (for the old and the new sequence) satisfy $b'_m = b_m + k \bmod (n+1)$. For $m = 1$ one has $b_m = a_1$, $b'_m = a'_1$, so the equation is proved. Suppose it to be true for all $i < m$; then the sets of free parking spots after $m-1$ cars for the new and for the old sequence differ by the cyclic shift by k positions (that is, by addition of k modulo $(n+1)$). The m-th car starts parking at $a'_m = a_m + k \bmod (n+1)$, and then all the parking process will be shifted k positions modulo $(n+1)$ (only relative positions of the free and the occupied spots matter!). Therefore $b'_m = b_m + k \bmod (n+1)$.

As a corollary, the only spot that remains free also undergoes a cyclic shift by k positions. $\square$

Lemma 3. *$F(a_1, \dots, a_n)$ will not change if the terms of the sequence $a_1, \dots, a_n$ are permuted.*

Proof. Suppose $F(a_1, \dots, a_n) = k$ and consider a sequence $a'_i = a_i + (n-k) \bmod (n+1)$. If the sequence $b_1, \dots, b_n$ is obtained from $a_1, \dots, a_n$ by permutation of terms then $b'_i \stackrel{\text{def}}{=} b_i + (n-k) \bmod (n+1)$ is obtained from a'_i by the same permutation. By Lemma 2 one has $F(a'_1, \dots, a'_n) = k + (n-k) = n$, therefore by Lemma 1 the sequence $a'_1, \dots, a'_n$ is a parking function. By Corollary 1 $b'_1, \dots, b'_n$ is a parking function, too, hence $F(b'_1, \dots, b'_n) = n$ by Lemma 1. By Lemma 2 one has $n = F(b'_1, \dots, b'_n) = F(b_1, \dots, b_n) + (n-k)$, so that $F(b_1, \dots, b_n) = k = F(a_1, \dots, a_n)$. $\square$

Lemma 2 implies now that for all $k = 0, \dots, n$ the sets $\{(a_1, \dots, a_n) \mid F(a_1, \dots, a_n) = k\}$ contain the same number of elements. Therefore, each of them, including the set $\{(a_1, \dots, a_n) \mid F(a_1, \dots, a_n) = n\}$ of parking functions (by Lemma 1) contains $1/(n+1)$ of the total number of sequences $a_1, \dots, a_n$. Each $a_i \in \{0, \dots, n\}$, the number of terms is n, so the total number of sequences is $(n+1)^n$, and the number of parking functions is $(n+1)^{n-1}$ — the first statement is proved. Similarly, the number of equivalence classes of the parking functions is $1/(n+1)$ of the number of equivalence classes of all sequences $a_1, \dots, a_n$. This number was computed at Problem 3d of the problem set 6–7: it is equal to $\binom{2n}{n}$. So the number of equivalence classes of parking functions is $\frac{1}{n+1}\binom{2n}{n}$ — the Catalan number. $\square$

Exercises

Let τ be a tree with $n+1$ vertices. Give every vertex a second number by the following rule (called "width-first" in literature; essentially it is the rule of royal succession): a vertex with the original number 1 (a "root", or a "patriarch", or a "dynasty founder") gets the second number 0. If this vertex is joined by edges with k vertices ("children") with the original numbers $p_1 < \dots < p_k$ then p_1 gets the second number 1, p_2 — the second number 2, etc. Then take p_1, let it be joined by edges with the root and ℓ more vertices (children) numbered $q_1 < \dots < q_\ell$; then q_1 gets the second number $k+1$, q_2, the second number $k+2$, etc. Then children of p_2 are numbered, etc.

Consider a sequence $a_1, \dots, a_n$ where a_i is the *second* number of the parent of the vertex with the *original* number $i+1$.

Exercise 1. Prove that $a_1, \dots, a_n$ is a parking function.

Exercise 2. Describe parking functions $a_1, \dots, a_n$ for a) "linear" trees (the vertex u_1 is joined with u_2, it is joined with u_3, etc.), b) monotonic trees, c) trees with k hanging vertices, not including the root (which may be, or may be not a hanging vertex).