# LECTURE 5.

ABSTRACT. Binomial series, exponentials and logarithms.

Let $\alpha$ be an arbitrary complex number, and let $k$ be a positive integer. The order $k$ binomial coefficient of $\alpha$ is defined as

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}.$$

Also take by definition $\binom{\alpha}{0} = 1$ for any $\alpha$.

**Problem 1.** a) Prove that if $\alpha = m$ is a nonnegative integer then $\binom{\alpha}{n} = \frac{m!}{n!(m-n)!}$ (so that the notation is consistent). b) Prove that $\binom{\alpha}{n} \neq 0$ for all $n$, except for the case when $\alpha$ is a nonnegative integer. c) Prove the Pascal's identity $\binom{\alpha}{n} = \binom{\alpha-1}{n} + \binom{\alpha-1}{n-1}$ for all $\alpha \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$.

The *binomial series* is defined as the power series

$$(1+t)^\alpha = 1 + \binom{\alpha}{1}t + \binom{\alpha}{2}t^2 + \binom{\alpha}{3}t^3 + \dots = \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n.$$

By Newton's formula, if $\alpha = m$ is a nonnegative integer then the series is actually a polynomial equal to $(1+t)^\alpha = (1+t)^m$. If $\alpha$ is not a nonnegative integer, then the binomial series is indeed an infinite power series.

*Remark.* We are not going to define $(x+y)^\alpha$ for two variables $x$ and $y$ and any $\alpha$, to avoid discussing what $x^\alpha$ would mean if $\alpha$ is not an integer. (On the other hand, $1^\alpha = 1$ for all $\alpha$.)

An *exponential* is a power series

$$\exp(t) = 1 + \frac{1}{1!}t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \dots = \sum_{n=0}^{\infty} \frac{t^n}{n!},$$

**Problem 2.** a) Prove that the exponential satisfies the differential equation $\exp'(t) = \exp(t)$. b) Prove that if a power series $f$ satisfies $f'(t) = f(t)$ then $f(t) = A\exp(t)$ for some constant $A$. c) Prove that the exponential satisfies the relation $\exp(t+s) = \exp(t)\exp(s)$. d) Prove that if a power series $f$ satisfies $f(t+s) = f(t)f(s)$ then $f(t) = \exp(at)$ for some constant $a$, or $f \equiv 0$.

A *logarithm* is defined as a power series

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \dots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{t^n}{n}.$$

**Problem 3.** a) Prove that $\ln'(1+t) = 1/(1+t)$. b) Prove that if a power series $f$ satisfies the differential equation $f'(t) = \alpha f(t)/(1+t)$ then $f(t) = A(1+t)^\alpha$ for some constant $A$. c) Prove the identity $\exp(\alpha \ln(1+t)) = (1+t)^\alpha$. d) Prove the identity $(1+t)^{\alpha+\beta} = (1+t)^\alpha(1+t)^\beta$.

**Problem 4.** a) Prove that $\ln((1+s)(1+t)) = \ln(1+s) + \ln(1+t)$. b) Prove that $\ln((1+t)^\alpha) = \alpha\ln(1+t)$.

Define the power series cos and sin by the equations $\cos t = \frac{1}{2}(\exp(it) + \exp(-it))$ and $\sin t = -\frac{i}{2}(\exp(it) - \exp(-it))$.

**Problem 5.** a) Write down the power series $\cos t$ and $\sin t$ explicitly. b) Prove the identities $\cos' t = -\sin t$ and $\sin' t = \cos t$. c) Prove the identity $\cos^2 t + \sin^2 t = 1$.

Define the power series $\arcsin t$ as

$$\arcsin t = t + \frac{1}{2\cdot 3}t^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}t^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}t^7 + \dots = t + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!\cdot(2n+1)} t^{2n+1}.$$

**Problem 6.** a) Prove that $\arcsin' t = (1-t^2)^{-1/2}$. b) Prove that $\sin \arcsin t = t$.