LECTURE 3.

ABSTRACT. Linear recursion II: explicit formulas.

Recall that a number λ is called a root of the polynomial P of multiplicity m if $P(\lambda) = P'(\lambda) = \cdots = P^{(m-1)}(\lambda) = 0$ and $P^{(m)}(\lambda) \neq 0$. A root of multiplicity 1 is called simple.

Theorem 1. *A sequence $x_0, x_1, \ldots$ satisfies a linear recursion*

$$x_{n+k} = a_0 x_n + a_1 x_{n+1} + \cdots + a_{k-1} x_{n+k-1} \tag{1}$$

if and only if $x_n = Q_1(n)\lambda_1^n + \cdots + Q_s(n)\lambda_s^n$. Here $\lambda_1, \ldots, \lambda_s$ are pairwise distinct roots of the polynomial $\chi(t) = t^k - a_{k-1}t^{k-1} - \cdots - a_0$, $Q_1, \ldots, Q_s$ are polynomials, and the degree of any polynomial Q_i does not exceed $m_i - 1$ where m_i is the multiplicity of λ_i ($i = 1, \ldots, s$).

Proof of the "if" part. Suppose λ to be a root of $\chi(t) = t^k - a_{k-1}t^{k-1} - \cdots - a_0$ of multiplicity m. By linearity (Theorem 1 of Lecture 2) it suffices to prove that the sequence $x_n = n^s \lambda^n$, where $0 \le s \le m-1$, satisfies (1).

Denote by D the linear operator $D = t\frac{d}{dt}$, so if $C = \sum_{n=0}^{\infty} c_n t^n$ is a power series then $DC = \sum_{n=0}^{\infty} n c_n t^n$. An easy induction shows that $D^k = \sum_{m=1}^{k} c_{km} t^m \frac{d^m}{dt^m}$ for some constants c_{km} — so, by definition of the multiplicity, $(D^k\chi)(\lambda) = 0$ for any $0 \le k \le m-1$. Then,

$$\begin{aligned} x_{n+k} - a_{k-1}x_{n+k-1} - \cdots - a_0 x_n &= (n+k)^s\lambda^{n+k} - a_{k-1}(n+k-1)^s\lambda^{n+k-1} - \cdots - a_0 n^s \lambda^n \\ &= \lambda^n \times \Bigg(n^s(\lambda^k - a_{k-1}\lambda^{k-1} - \cdots - a_0) + n^{s-1}\binom{s}{1}(k\lambda^k - (k-1)a_{k-1}\lambda^{k-1} - \cdots - 1\cdot a_1\lambda - 0\cdot a_0) \\ &\qquad - \cdots - n^0\binom{s}{s}(k^s\lambda^k - (k-1)^s a_{k-1}\lambda^k - \cdots - 1^s\cdot a_1\lambda - 0^s\cdot a_0)\Bigg) \\ &= \lambda^n\left(n^s\chi(\lambda) + \binom{s}{1}n^{s-1}(D\chi)(\lambda) + \cdots + \binom{s}{s}n^0(D^s\chi)(\lambda)\right) = 0 \end{aligned}$$

$\square$

To prove the "only if" part of the theorem we will need three lemmas:

Lemma 1. *For all power series A and B one has $\frac{d^n}{dt^n}(AB) = \sum_{k=0}^{n}\binom{n}{k}\frac{d^kA}{dt^k}\cdot\frac{d^{n-k}B}{dt^{n-k}}$.*

Proof. Easy induction similar to the proof of the binomial formula. $\square$

Lemma 2 (generalized Bezout theorem)**.** *λ is a root of multiplicity m of the polynomial P if and only if $P(t) = (t-\lambda)^m Q(t)$ where $Q(\lambda) \neq 0$.*

Proof. Divide $P(t)$ by $(t-\lambda)$ with a residue: $P(t) = (t-\lambda)P_1(t) + a$. The degree of the residue a is 0, that is, a is a constant. Substituting $t = \lambda$ gives $a = P(\lambda)$. Thus, P is divisible by $(t-\lambda)$ if and only if $P(\lambda) = 0$. (This statement is the classical Bezout theorem.)

Proceed by induction by m. Suppose the lemma is proved for multiplicities smaller than m, and λ is a root of P of the multuplicity m. Hence $P(t) = (t-\lambda)^{m-1}P_1(t)$. Taking derivative one obtains $P'(t) = (m-1)(t-\lambda)^{m-2}P_1(t) + (t-\lambda)^{m-1}P_1'(t)$. By the definition λ is a root of multiplicity $m-1$ of P', so by the induction hypothesis $P'(t) = (t-\lambda)^{m-1}P_2(t)$ for some P_2, and therefore $P_1(t)$ is divisible by $(t-\lambda)$: $P(t) = (t-\lambda)Q(t)$. So, $P(t) = (t-\lambda)^m Q(t)$. By Lemma 1 one has $P^{(m)}(\lambda) = m!Q(\lambda) \neq 0$, hence $Q(\lambda) \neq 0$.

The other way round, let $P(t) = (t-\lambda)^m Q(t)$ where $Q(\lambda) \neq 0$. Lemma 1 implies that $P^{(s)}(\lambda) = 0$ for any $0 \le s \le m-1$, and $P^{(m)}(\lambda) = m!Q(\lambda) \neq 0$, so λ is a root of multiplicity m. $\square$

Lemma 3. *If $\lambda_1, \ldots, \lambda_s$ be pairwise distinct numbers and $m_1, \ldots, m_s$ are positive integers. Then for any constants $u_0^{(1)}, \ldots, u_{m_1-1}^{(1)}, \ldots, u_0^{(s)}, \ldots, u_{m_s-1}^{(s)}$ there exists a unique polynomial Q of degree less than $m_1 + \cdots + m_s$ such that $Q^{(j)}(\lambda_k) = u_k^{(j)}$ for all k and j.*

Proof. Uniqueness: let Q_1, Q_2 be two polynomials satisfying the conditions of the lemma. Then $Q \stackrel{\text{def}}{=} Q_1 - Q_2$ is a polynomial with $Q^{(j)}(\lambda_k) = 0$ for any $k = 1, \ldots, s$ and any $j = 1, \ldots, m_k$. It means that for all $k = 1, \ldots, s$ the number λ_k is a root of Q of multiplicity m_k or more. By Lemma 2 the polynomial $Q(t)$ is divisible by $(t-\lambda_1)^{m_1}\ldots(t-\lambda_s)^{m_s}$. Once the degree of Q is less than that of $(t-\lambda_1)^{m_1}\ldots(t-\lambda_s)^{m_s}$, it means that $Q = 0$.

Existence: see Exercise 2. $\square$

Lemma 4. *Let A and B be polynomials, and $\lambda_1, \dots, \lambda_s$ are the pairwise distinct roots of A of multiplicities $m_1, \dots, m_s$. Then there exists unique polynomials $P_0, P_1, \dots, P_s$ such that $\frac{B(t)}{A(t)} = P_0(t) + \frac{P_1(t)}{(t-\lambda_1)^{m_1}} + \dots + \frac{P_s(t)}{(t-\lambda_s)^{m_s}}$ and the degree of every P_i, $1 \le i \le s$, is less or equal to $m_i - 1$.*

Proof. First, divide B by A: $B(t) = P_0(t)A(t) + R(t)$, where R is the residue: $\deg R < \deg A$. So, the lemma is reduced to the case when $\deg B < \deg A$ (and $P_0 = 0$).

By the principal theorem of algebra and Lemma 2 one has $A(t) = C(t-\lambda_1)^{m_1} \dots (t-\lambda_s)^{m_s}$ for some $C =$ *const.* $\neq 0$. Thus the statement of the lemma is equivalent to $B(t)/C = P_1(t)R_1(t) + \dots + P_s(t)R_s(t)$ where $R_k(t) \stackrel{\text{def}}{=} (t-\lambda_1)^{m_1} + \dots + \widehat{(t-\lambda_k)^{m_k}} + \dots + (t-\lambda_s)^{m_s}$, $k = 1, \dots, s$.

By Lemma 2 one has $R_k^{(p)}(\lambda_q) = 0$ if $q \neq k$ and $0 \le p \le m_q$. By Lemma 1 if $k < m_i$ then one has $B^{(k)}(\lambda_i) = P_i^{(k)}(\lambda_i) \cdot (\lambda_i - \lambda_1)^{m_1} \dots (\lambda_i - \lambda_s)^{m_s} + \dots$ where the dots stand for a linear combination of $P_i^{(\ell)}(\lambda_i)$ with $\ell < k$. So, by induction, we see that the values of $P_i(\lambda_i), P'(\lambda_i), \dots, P^{(m_i-1)}(\lambda_i)$ are uniquely defined. By Taylor's formula these conditions define uniquely a polynomial P_i of degree $m_i - 1$ or less. □

Proof of the "only if" part. Take $X = \sum_{n=0}^{\infty} x_n t^n$; by Theorem 2 of Lecture 2 and Lemma 4 one has $X = B(t)/A(t) = P_0(t) + \frac{P_1(t)}{(t-\mu_1)^{m_1}} + \dots + \frac{P_s(t)}{(t-\mu_s)^{m_s}}$. It is easy to see that all $\mu_i \neq 0$. By the binomial formula $(t-\mu)^{-m} = \sum_{n=0}^{\infty} \mu^{-m-n} \frac{m(m-1)\dots(m-n+1)}{n!} t^n = \lambda^m \sum_{n-0}^{\infty} \lambda^n Q(n) t^n$ where $\lambda = 1/\mu$ and Q is a polynomial of degree m. The same is true for the fraction $\frac{t^k}{(t-\mu)^m}$ with any k, and therefore for a fraction $\frac{P(t)}{(t-\mu)^m}$ with any polynomial P. So, x_n is like in the theorem. □

Exercises

Exercise 1. Find a direct formula for the constants c_{km} in the equality $(t\frac{d}{dt})^k = \sum_{m=0}^{k} c_{km} t^m \frac{d^m}{dt^m}$.

Hint. One has $\frac{d}{dt} t = t\frac{d}{dt} + 1$ by the chain rule. Then one has, for example, $(t\frac{d}{dt})^2 = t(t\frac{d}{dt} + 1)\frac{d}{dt} = t^2 \frac{d^2}{dt^2} + t\frac{d}{dt}$, so that $c_{22} = c_{21} = 1$.

Exercise 2. a) Let $\lambda_1, \dots, \lambda_{n+1} \in \mathbb{C}$ be pairwise distinct numbers. Find a polynomial Q of degree n such that $Q(\lambda_1) = 1$, $Q(\lambda_2) = \dots = Q(\lambda_{n+1}) = 0$. b) Let $u_1, \dots, u_{n+1} \in \mathbb{C}$ be any constants. Find a polynomial of degree n or less such that $Q(\lambda_1) = u_1, \dots, Q(\lambda_{n+1}) = u_{n+1}$. c) Find a polynomial Q_1 of degree n such that $Q(\lambda_1) = \dots = Q(\lambda_n) = 0$ (note the last subscript!) and $Q'(\lambda_1) = 1$. d) Find a polynomial R of degree n such that $R(\lambda_2) = \dots = R(\lambda_n) = 0$ (note the first and the last subscripts!) and $R(\lambda_1) = 1$. Then find a polynomial Q of degree n or less satisfying the same equalities and additionally such that $Q'(\lambda_1) = 0$. e) Let $u_1^{(0)}, u_1^{(1)}, u_2, \dots, u_n$ be any constants. Prove that there exists a polynomial Q of degree n or less such that $Q(\lambda) = u_1^{(0)}$, $Q'(\lambda_1) = u_1^{(1)}$, $Q(\lambda_k) = u_k$ for all $k = 2, \dots, n$. f) Prove existence in Lemma 3.

Exercise 3. Let $X(t) \stackrel{\text{def}}{=} x_0 + x_1 t + x_2 t^2 + \dots$ (a generating function) where the sequence x_n satisfies: a) $x_{x+2} = 5x_{n+1} - 6x_n$, $x_0 = 2, x_1 = 1$; b) $x_{n+2} = 2x_{n+1} - x_n$, $x_0 = 1, x_1 = 3$. Find $X(t)$ as a rational function $A(t)/B(t)$, represent it as a sum of elementary fractions, and develop it into power series to obtain an explicit formula for the sequence x_n.

Exercise 4. How many sequences $a_1, \dots, a_n$ of zeros and ones do not contain three ones in a row?

Exercise 5. a) Prove that the series $f(t) = \sum_{n=0}^{\infty} n! t^n$ diverges for any $t \neq 0$. b) Prove that f satisfies the differential equation $t^2 f' + (t-1)f + 1 = 0$. c*) Find all the power series satisfying this equation. Is there a power series satisfying this equation and convergent for some $t \neq 0$?