LECTURE 2.

ABSTRACT. Linear recursion I: generating functions.

Definition 1. A sequence $x = (x_0, x_1, x_2, \dots)$ is said to satisfy a k-th order linear recursion with constant coefficients $a_0, \dots, a_{k-1}$ if for every $n \ge 0$ there is a relation $x_{n+k} = a_0 x_n + a_1 x_{n+1} + \dots + a_{k-1} x_{n+k-1}$.

Example 1. A sequence satisfying a first order recursion $x_{n+1} = qx_n$ (here $a_0 = q$) is called a geometric progression.

Example 2. A Fibonacci sequence $0, 1, 1, 2, 3, 5, \dots$ satisfies a second order recursion $x_{n+2} = x_n + x_{n+1}$; here $a_0 = a_1 = 1$.

Example 3. A periodic sequence $x = (x_0, x_1, \dots, x_{k-1}, x_0, x_1, \dots)$ with a period of length k satisfies a k-th order recursion $x_{n+k} = x_n$. Here $a_0 = 1$ and $a_1 = \dots = a_{k-1} = 0$. In particular, a constant sequence $x_n = a$ satisfies a first order recursion $x_{n+1} = x_n$. Here $k = 1$ and $a_0 = 1$.

Example 4. A sequence $x_n = n$ of natural numbers satisfies the second order recursion $x_{n+2} = 2x_{n+1} - x_n$.

Example 5. If a sequence x satisfies a linear recursion $x_{n+k} = a_0 x_n + a_1 x_{n+1} + \dots + a_{k-1} x_{n+k-1}$ of order k then the sequence y defined as $y_n \stackrel{\text{def}}{=} x_{n-1}$ for $n \ge 1$ and y_0 is arbitrary (i.e. y is obtained from x by insertion of an arbitrary number y_0 to the 0-th position), satisfies a linear recursion $y_{n+k+1} = a_0 x_{n+1} + a_1 x_{n+2} + \dots + a_{k-1} x_{n+k}$ of order $k+1$. Repeating this procedure, one arrives to the conclusion that the linear recursion is not sensitive to any finite initial segment of a sequence.

Theorem 1.
(1) *If sequences $x = (x_0, x_1, \dots)$ and $y = (y_0, y_1, \dots)$ satisfy a k-th order linear recursion with coefficients $a_0, \dots, a_{k-1}$, and c is a constant, then the sequences $x + y \stackrel{def}{=} (x_0 + y_0, x_1 + y_1, \dots)$ and $cx \stackrel{def}{=} cx_0, cx_1, \dots$ satisfy the same recursion.*
(2) *For any constants $a_0, a_1, \dots, a_{k-1}$ and $q_0, q_1, \dots, q_{k-1}$ there exists exactly one sequence $x = (x_0, x_1, \dots)$ such that $x_0 = q_0, \dots, x_{k-1} = q_{k-1}$ and the sequence satisfies the k-th order linear recursion with the coefficients $a_0, \dots, a_{k-1}$.*

The proof is an exercise. Another formulation of Theorem 1:

Theorem 1, second formulation. *For all $a_0, \dots, a_{k-1}$ the sequences $x_0, x_1, \dots$ satisfying the k-th order linear recursion with coefficients $a_0, \dots, a_{k-1}$ form a linear space V. The map $A : V \to \mathbb{C}^k$ defined as $A(x) = (x_0, \dots, x_{k-1})$ is a linear isomorphism (and therefore $\dim V = k$).*

Theorem 2. *A sequence $x = (x_0, x_1, \dots)$ satisfies a linear recursion if and only if the power series $X = \sum_{n=0}^{\infty} x_n t^n$ is a rational function (i.e. $X = B(t)/A(t)$ where B and A are polynomials).*

Proof. Let x satisfy a recursion. Take the recursion relation $x_{n+k} = a_0 x_n + a_1 x_{n+1} + \dots + a_{k-1} x_{n+k-1}$, multiply it by t^{n+k} and sum over $n = 0$ to ∞, to obtain:

$$\sum_{n=0}^{\infty} x_{n+k} t^{n+k} = \sum_{s=0}^{k-1} a_s \sum_{n=0}^{\infty} x_{n+s} t^{n+s} = \sum_{s=0}^{k} a_s t^{k-s} \sum_{n=0}^{\infty} x_{n+s} t^{n+s} \tag{1}$$

For any number s one has $\sum_{n=0}^{\infty} x_{n+s} t^{n+s} = \sum_{n=s}^{\infty} x_n t^n = X - P_s(t)$ where $P_s(t) \stackrel{\text{def}}{=} x_0 + \dots + x_{s-1} t^{s-1}$. So equation (1) reads as

$$X - P_n(t) = \sum_{s=0}^{k-1} a_s t^{k-s} (X - P_s(t))$$

implying

$$(1 - \sum_{s=0}^{k-1} a_s t^{k-s}) X = P_n(t) - \sum_{s=0}^{k-1} a_s t^{k-s} P_k(t),$$

so $X = B(t)/A(t)$ where $A(t) \stackrel{\text{def}}{=} 1 - a_{k-1} t - \dots - a_0 t^k$ and $B(t) = P_k(t) - a_{k-1} t P_{k-1}(t) - \dots - a_1 t^{k-1} P_1(t)$ (for correctness take $P_0 \stackrel{\text{def}}{=} 0$). The degree of A is k (the degree of the recursion) or less.

Let now $X(t) = B(t)/A(t)$ where $X(t) = \sum_{n=0}^{\infty} x_n t^n$, $A(t) = a_0 + \dots + a_k t^k$ and $B(t) = b_0 + \dots + b_m t^m$. Without loss of generality one may assume that $a_k \ne 0$ (else we just consider A a polynomial of degree less than

k). Then $B = AX = \sum_{n=0}^{\infty}\left(\sum_{s=0}^{k} a_s x_{n-s}\right)t^n$ has no terms of degrees $m+1$ and more; thus, $\sum_{s=0}^{k} a_s x_{n-s} = 0$ for $n \geq m+1$, which is equivalent to $x_{n+m+1+k} = -a_0/a_k \cdot x_{n+m+1} - \cdots - a_{k-1}/a_k \cdot x_{n+m+k}$ for all $n \geq 0$ — it is a linear recursion of the order $(m+1+k)$. □

Example 6. *Fibonacci numbers* is a sequence of integers $x_0, x_1, \dots$ such that $x_0 = 0$, $x_1 = 1$ and $x_n = x_{n-1} + x_{n-2}$ for all $n \geq 2$. Let $X = \sum_{n=0}^{\infty} x_n t^n$. Then $(t^2+t-1)X = \sum_{n=0}^{\infty} x_n(t^{n+2} + t^{n+1} - t^n) = -x_0 + (x_0 - x_1)t + \sum_{n=2}^{\infty}(-x_n + x_{n-1} + x_{n-2})t^n = -t$, so $X = \frac{t}{1-t-t^2}$.

The numbers $\varphi_1 = (-1+\sqrt{5})/2$ and $\varphi_2 = -(1+\sqrt{5})/2$ are roots of the polynomial t^2+t-1, so $t^2 + t - 1 = (t-\varphi_1)(t-\varphi_2)$. Therefore one has $\frac{t}{1-t-t^2} = \frac{1}{\varphi_2-\varphi_1}\left(\frac{\varphi_1}{t-\varphi_1} - \frac{\varphi_2}{t-\varphi_2}\right) = \frac{1}{\varphi_1-\varphi_2}\left(\frac{1}{1-t/\varphi_1} - \frac{1}{1-t/\varphi_2}\right) = \frac{1}{\varphi_1-\varphi_2}\sum_{n=0}^{\infty}\left(\frac{1}{\varphi_1^n} - \frac{1}{\varphi_2^n}\right)t^n$. Once $1/\varphi_1 = (1+\sqrt{5})/2$ and $1/\varphi_2 = (1-\sqrt{5})/2$ and $\varphi_1 - \varphi_2 = \sqrt{5}$, one obtains an explicit formula for the Fibonacci numbers:

$$x_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

In other words, $x_n = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n\left(1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n\right)$. Since $\left|\frac{1-\sqrt{5}}{1+\sqrt{5}}\right| < 1$, one has $y_n \stackrel{\text{def}}{=} \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n \to 0$ as $n \to \infty$. Therefore $\lim_{n\to\infty} x_n/x_{n-1} = \frac{1+\sqrt{5}}{2}\lim_{n\to\infty}\frac{1+y_n}{1+y_{n-1}} = \frac{1+\sqrt{5}}{2}$ (the "golden ratio").

Exercises

Exercise 1. Make the formulation of Theorem 2 more precise: how is the denominator $A(t)$ connected with the recursion equation?

Exercise 2. For the Fibonacci numbers x_n one has $\sum_{n=0}^{\infty} x_n t^n = \frac{t}{1-t-t^2} = t\sum_{k=0}^{\infty}(t+t^2)^k = \sum_{k=0}^{\infty} t^{k+1}(1+t)^k$. Use the binomial formula $(1+t)^k = \sum_{m=0}^{k}\binom{k}{m}t^m$ to obtain an expression of x_n via binomial coefficients.

Exercise 3. Let x_n be the number of ways to tile a rectangle $3 \times n$ by rectangles 1×2 (positioned horizontally or vertically). Prove that x_n satisfies a linear recursion. What is the order of this recursion? Find an explicit formula for x_n.