LECTURE 1.

ABSTRACT. Binomial coefficients.

Let n, m be nonnegative integers, $0 \le m \le n$.

Definition 1. $\binom{n}{m}$ is the number of subsets of m elements in the set $1, 2, \dots, n$ (or in any other set of n elements).

Definition 2. $\binom{n}{m}$ is the coefficient at t^m in the polynomial $(1+t)^n$. In other words,

$$(1+t)^n = \binom{n}{0} + \binom{n}{1}t + \dots + \binom{n}{m}t^m + \dots + \binom{n}{n}t^n. \tag{1}$$

Pascal's triangle is a diagram of integers organized as follows:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

The leftmost and the rightmost terms of each row are equal to 1, and every term inside the row is a sum of its two neighbors from the upper row.

Definition 3. $\binom{n}{m}$ is the m-th term in the n-th row (numeration starting from 0) of the Pascal's triangle.

Definition 4. $\binom{n}{m} \stackrel{\text{def}}{=} \frac{n!}{m!(n-m)!}$ where $k!$ is defined as a product $1 \times 2 \times \dots \times k$ for a positive integer k and $0! = 1$ by definition.

Theorem 1. *All the four definitions above are equivalent. The numbers $\binom{n}{m}$ so defined have the following properties:*

(1) *(Pascal's identity)* $\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}$,
(2) $\binom{n}{m} = \binom{n}{n-m}$;
(3) $\sum_{m=0}^{n} \binom{n}{m} \stackrel{def}{=} \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$;
(4) $\sum_{m=0}^{n} (-1)^m \binom{n}{m} \stackrel{def}{=} \binom{n}{0} - \binom{n}{1} + \dots \pm \binom{n}{n} = 0$ *if* $n > 0$;
(5) $\sum_{m=0}^{n} m\binom{n}{m} \stackrel{def}{=} \binom{n}{1} + 2\binom{n}{2} + \dots + n\binom{n}{n} = n2^{n-1}$;
(6) $\sum_{m=0}^{n} \binom{n}{m}^2 \stackrel{def}{=} \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$.

Proof.

1 $\Longleftrightarrow$ 2. One has $(1+t)^n = (1+t)(1+t)\dots(1+t)$ (n terms); number the factors left to right from 1 to n. Opening brackets one has to choose either 1 or t from each of the n factors. Let $A \subset \{1, 2, \dots, n\}$ be the set of numbers of factors from which t was chosen. Such choice gives the term $t^{\#A}$ where $\#A$ means the number of elements in A. Now the coefficient at t^m equals the numbers of $A \subset \{1, 2, \dots, n\}$ such that $\#A = m$.

1 $\Longleftrightarrow$ 4. Arrange all the elements $1, 2, \dots, n$ in some way: $\sigma(1), \sigma(2), \dots, \sigma(n)$ where $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is a bijection, and consider a m-element set $\{\sigma(1), \dots, \sigma(m)\}$. There are totally $n!$ arrangements (bijections) σ. A permutation of the first m elements of the arrangement and of its last $(n-m)$ elements will not change the set $\{\sigma(1), \dots, \sigma(m)\}$, so a given set $\{a_1, \dots, a_m\}$ can be obtained from $m!(n-m)!$ permutations. Therefore the total number of sets is $\frac{n!}{m!(n-m)!}$.

2 $\Longleftrightarrow$ 4. The m-th derivative of the two sides of the identity $(1+t)^n = \sum_{k=0}^{k} \binom{n}{k} t^k$ is $n(n-1)\dots(n-m+1)(1+t)^{n-m} = \sum_{k=m}^{n} \binom{n}{k} k(k-1)\dots(k-m+1)t^{k-m}$. Taking $t = 0$ we see that all the terms in the right-hand side, except the very first one, disappear. So $t = 0$ gives $n(n-1)\dots(n-m+1) = \frac{n!}{(n-m)!} = m!\binom{n}{m}$.

To prove that 3 is equivalent to the other ones it suffices to prove Property 1 for them.

Property 1. For Definition 2: the free term in $(1+t)^n$ is equal to the value of $(1+t)^n$ at $t=0$, that is, to 1. It means $\binom{n}{0}=1$. The leading term in $(1+t)^n$ is, obviously, t^n, which means $\binom{n}{n}=1$. Consider now the formula $(1+t)^n=(1+t)\cdot(1+t)^{n-1}$. The term $\binom{n}{m}t^m$ in the left-hand side is the sum of two terms in the right-hand side: $1\cdot\binom{n-1}{m}t^m$ and $t\cdot\binom{n-1}{m-1}t^{m-1}$. Thus one has $\binom{n}{m}=\binom{n-1}{m-1}+\binom{n-1}{m}$ — it is Property 1.

For Definition 4: denote $b_m^n \stackrel{\text{def}}{=} \frac{n!}{m!(n-m)!}$ Now $b_0^n=\frac{n!}{n!0!}=1=\frac{n!}{0!n!}=b_n^n$. Also

$$b_{m-1}^{n-1}+b_m^{n-1}=\frac{(n-1)!}{(m-1)!(n-m)!}+\frac{(n-1)!}{m!(n-m-1)!}=\frac{(n-1)!}{m!(n-m)!}(m+n-m)=\frac{n\cdot(n-1)!}{m!(n-m)!}=b_m^n.$$

□

Remark. It is often convenient to assume $\binom{n}{m}=0$ if $m<0$ or $m>n$. This agrees with Definitions 2 and 3 (check!).

For Definition 1: split the set of all m-element subsets of $\{1,\dots,n\}$ into two groups: the subsets containing n and the subsets not containing n. If a subset does not contain n then it is a m-element subset of $1,\dots,(n-1)$; there are $\binom{n-1}{m}$ of them. If a subset contains n then its remaining elements form a $(m-1)$-element subset of $1,\dots,(n-1)$, so there are $\binom{n-1}{m-1}$ of them. The total number of all subsets is $\binom{n-1}{m}+\binom{n-1}{m-1}$; but from the other side, it is $\binom{n}{m}$.

Property 2. For Definition 2: one has $(1+t)^n=t^n\cdot(1+1/t)^n$. The $\binom{n}{m}t^m$ in the left-hand side is equal in the right-hand side to t^n multiplied by the term containing $\frac{1}{t^{n-m}}$. This term is $\binom{n}{n-m}\frac{1}{t^{n-m}}$, so $\binom{n}{m}=\binom{n}{n-m}$.

For Definition 1: to any m-element subset $A\subseteq\{1,\dots,n\}$ one can relate its complement $\overline{A}\stackrel{\text{def}}{=}\{1,\dots,n\}\setminus A$; the complement contains $n-m$ elements. A itself is determined by its complement (it is the complement of the complement), so $\{1,\dots,n\}$ has as many m-element subsets as $(n-m)$-element subsets.

For Definition 4: evident. For Definition 3: trivial induction by the number of the row.

Property 3. For Definition 2: substitute $t=1$ in the indentity $(1+t)^n=\binom{n}{0}+\binom{n}{1}t+\dots+\binom{n}{m}t^m+\dots+\binom{n}{n}t^n$.

For Definition 3: it follows from the definition that the sum of the elements of the n-th row is twice the sum of the elements of the $(n-1)$-th row. Since the sum of the elements of the 0-th row is $1=2^0$, trivial induction implies that it is 2^n in the n-th row.

For Definition 1: to describe a subset of $\{1,\dots,n\}$ one should say for every its element whether it belongs to a subset or not. So there are n independent choices from 2 variants each; this gives the total number of subsets being 2^n. Since the number of m-element subsets is $\binom{n}{m}$ where m can be from 0 to n, Property 3 follows.

For Definition 4 no proof is known.

Property 4. For Definition 2: substitute $t=-1$ at $(1+t)^n=\sum_{m=0}^n\binom{n}{m}t^m$.

For Definition 3: the alternating-sign sum of the elements of the n-th row is equal twice the alternating sum of the elements of the $(n-1)$-th row, provided $n>0$ (check!). Now induction on n implies Property 4.

For Definition 1: let $A\subset\{1,\dots,n\}$; denote $c(A)\stackrel{\text{def}}{=}A\,\Delta\,\{n\}$ (symmetric difference, i.e. if A contains n then $c(A)$ is obtained by deletion of n, and if A does not contain n then $c(A)$ is obtained by adding it). The operation c is an involution: $c(c(A))=A$, the subsets of $\{1,\dots,n\}$ are split into nonintersecting pairs $\{A,c(A)\}$. The number of elements in A and $c(A)$ differs by 1 and therefore has opposite parity. So one obtains that $\{1,\dots,n\}$ has as many subsets A with $\#A$ even, as subsets A with $\#A$ odd. This implies $\sum_{k=0}^{n/2}\binom{n}{2k}=\sum_{k=0}^{n/2}\binom{n}{2k+1}$, which is equivalent to Property 4.

For Definition 4 no proof is known.

Property 5. For Definition 2: take a derivative of the identity (1) obtaining $n(1+t)^{n-1}=\sum_{m=0}^n m\binom{n}{m}t^{m-1}$. Substitution $t=1$ gives the required result.

For Definition 3: $\sum_{m=0}^n m\binom{n}{m}=\sum_{m=0}^n m\binom{n-1}{m}+\sum_{m=0}^n m\binom{n-1}{m-1}=\sum_{m=0}^n m\binom{n-1}{m}+\sum_{m=0}^n(m-1)\binom{n-1}{m-1}+\sum_{m=0}^n\binom{n-1}{m-1}$. The first two terms are equal to $(n-1)2^{n-2}$ by induction (note that the summation range here is irrelevant!), and the third one, to 2^{n-1} as proved above. Totally one has $(n-1)2^{n-2}+(n-1)2^{n-2}+2^{n-1}=n2^{n-1}$.

For Definition 1: $m\binom{n}{m}$ is the number of m-element subsets of $\{1,\dots,n\}$ with one distinguished element. Thus the left-hand side of the Property 5 is equal to the total number of subsets with one distinguished elements. Now take an element to distinguish — this gives n variants of choice — and then take a set (of any size) of the remaining elements, which give 2^{n-1} variants. Totally, $n2^{n-1}$.

Property 6. From Definition 2: consider the identity $(1+t)^n\cdot(1+t)^n=(1+t)^{2n}$. Replace the right-hand side and each of the factors in the left-hand side by the right-hand side of (1):

$$(\binom{n}{0}+\binom{n}{1}t+\dots+\binom{n}{n}t^n)\cdot(\binom{n}{0}+\binom{n}{1}t+\dots+\binom{n}{n}t^n)=\binom{2n}{0}+\binom{2n}{1}t+\dots+\binom{2n}{2n}t^{2n}.$$

The right-hand side contains the term $\binom{2n}{n}t^n$. The same power of t in the left-hand side appears as the result of multiplication of the terms $\binom{n}{m}t^m$ and $\binom{n}{n-m}t^{n-m}$ where m can be any number from 0 to n. Therefore one has $\binom{2n}{n} = \sum_{m=0}^{n} \binom{n}{m}\binom{n}{n-m} = \sum_{m=0}^{n} \binom{n}{m}^2$ — the last equality follows from Property 2.

From Definition 1: consider any set $A \subset \{1, \dots, 2n\}$. One has $A = A_1 \cup A_2$ where $A_1 = A \cap \{1, \dots, n\}$ and $A_2 = A \cap \{n, \dots, 2n\}$. The set A is uniquely determined by A_1 and A_2, which can be any sets containing $m \stackrel{\text{def}}{=} \#A_1$ and $n - m = \#A_2$ elements. There are $\binom{n}{m}$ possible sets A_1 and $\binom{n}{n-m} = \binom{n}{m}$ possible sets A_2. Thus there are $\binom{n}{m}^2$ possibilities for A with a fixed m, and Property 6 follows.

EXERCISES

Problem 1. a) Write the first 10 lines of the Pascal's triangle (numbered 0 to 9) modulo 2, that is, replacing even binomial coefficients by 0, and odd, by 1. b) Prove that if $n = 2^k$ for some integer k then the n-th line of Pascal's triangle modulo 2 looks like $10\dots01$ (all the terms inside are 0). c) Prove that if $n \neq 2^k$ then it is not so: the n-th line contains at least three 1s.

Problem 2. a) Compute for every n the sums $\sum_{m=0}^{\infty} \binom{n}{2m} = \binom{n}{0} + \binom{n}{2} + \dots + \binom{n}{2m} + \dots$ and $\sum_{m=0}^{\infty} \binom{n}{2m+1} = \binom{n}{1} + \binom{n}{3} + \dots + \binom{n}{2m+1} + \dots$. Give as many proofs as you can. (Recall that $\binom{n}{m} = 0$ if $m > n$, so the sums are actually finite.) b*) Compute $\sum_{m=0}^{\infty} \binom{n}{4m}$ and $\sum_{m=0}^{\infty} \binom{n}{3m}$.