**5.1.** Let X be a topological vector space, let  $X_0 \subset X$  be a vector subspace, and let  $\pi: X \to X/X_0$  denote the quotient map. Show that

(a) if  $\mathscr{U}$  is a neighborhood base at 0 in X, then  $\{\pi(U) : U \in \mathscr{U}\}$  is a neighborhood base at 0 in  $X/X_0$ ;

(b) the quotient  $X/X_0$  is Hausdorff if and only if  $X_0$  is closed in X.

**5.2.** Let X be a locally convex space, P be a directed fundamental family of seminorms on X, and  $X_0$  be a vector subspace of X. Show that the family  $\hat{P} = \{\hat{p} : p \in P\}$  of quotient seminorms is a fundamental family on  $X/X_0$ .

**5.3.** Let X be a topological vector space, let p be a continuous seminorm on X, and let  $\hat{p}$  denote the quotient seminorm on  $X/\overline{\{0\}}$ . Show that  $\hat{p}(x + \overline{\{0\}}) = p(x)$   $(x \in X)$ .

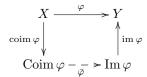
**5.4.** (a) Show that the kernel of a morphism  $\varphi \colon X \to Y$  in the category LCS of all locally convex spaces is the subspace  $\varphi^{-1}(0)$ , and that the cokernel of  $\varphi$  is the quotient  $Y/\varphi(X)$ .

(b) Describe kernels and cokernels of morphisms in the category HLCS of all Hausdorff locally convex spaces.

**5.5.** (a) Show that a morphism  $\varphi$  in LCS is a kernel if and only if it is topologically injective, and that  $\varphi$  is a cokernel if and only if it is open.

(b) Obtain a similar characterization of kernels and cokernels in HLCS.

Let  $\mathscr{A}$  be a category having a zero object. Suppose that each morphism in  $\mathscr{A}$  has a kernel and a cokernel. We define the *image* (Im  $\varphi$ , im  $\varphi$ ) of a morphism  $\varphi$  in  $\mathscr{A}$  to be the kernel of the cokernel of  $\varphi$ , and the *coimage* (Coim  $\varphi$ , coim  $\varphi$ ) of  $\varphi$  to be the cokernel of the kernel of  $\varphi$ . Thus for each  $\varphi \colon X \to Y$  there is a unique  $\overline{\varphi} \colon$  Coim  $\varphi \to \text{Im } \varphi$  making the following diagram commute:



We say that  $\varphi$  is *strict* if  $\overline{\varphi}$  is an isomorphism.

**5.6.** (a) Describe the image and the coimage of each morphism in the categories LCS and HLCS. (b) Show that a morphism  $\varphi \colon X \to Y$  in LCS is strict in the above sense if and only if it is strict as a continuous linear map (see the lectures), i.e., if and only if  $\varphi$  is an open map of X onto  $\varphi(X)$ . (c) Describe strict morphisms in HLCS.

**5.7.** Let X be a vector space equipped with the projective locally convex topology generated by a family of linear maps  $(\varphi_i \colon X \to X_i)_{i \in I}$ , where  $(X_i)_{i \in I}$  is a family of locally convex spaces. Show that (a) the projective topology on X is the weakest locally convex topology on X that makes all the maps  $\varphi_i$  continuous;

(b) the projective topology on X is the weakest topology on X that makes all the maps  $\varphi_i$  continuous;

(c) the projective topology on X is a unique locally convex topology on X having the following property: if Y is a locally convex space, then a linear map  $\psi: Y \to X$  is continuous if and only if all the maps  $\varphi_i \circ \psi: Y \to X_i$  are continuous;

(d) if  $\sigma_i$  is a neighborhood subbase at 0 in  $X_i$   $(i \in I)$ , then the family  $\{\varphi_i^{-1}(U_i) : U_i \in \sigma_i, i \in I\}$  is a neighborhood subbase at 0 in X.