

- 4.1.** Show that a relatively compact subset of a topological vector space is bounded.
- 4.2\***. Prove that a Hausdorff topological vector space is locally compact if and only if it is finite-dimensional. (This result was proved at the lectures in the special case of locally convex spaces.)
- 4.3.** Construct a linear map  $T$  between locally convex spaces  $X$  and  $Y$  which takes bounded subsets of  $X$  to bounded subsets of  $Y$ , but which is not sequentially continuous. Using this, find an example of a nonbornological locally convex space.
- 4.4\***. Construct a discontinuous, sequentially continuous linear map between locally convex spaces.
- 4.5.** Let  $\mathbf{LCS}$  denote the category of locally convex spaces, and let  $\mathbf{CBorn}$  denote the category of convex bornological spaces. Show that the functors  $\mathbf{vN}: \mathbf{LCS} \rightarrow \mathbf{CBorn}$  and  $\mathbf{top}: \mathbf{CBorn} \rightarrow \mathbf{LCS}$  are indeed functors (see the lecture), and construct a natural isomorphism

$$\mathrm{Hom}_{\mathbf{LCS}}(\mathbf{top}(X), Y) \cong \mathrm{Hom}_{\mathbf{CBorn}}(X, \mathbf{vN}(Y)) \quad (X \in \mathbf{CBorn}, Y \in \mathbf{LCS})$$

(in other words,  $(\mathbf{top}, \mathbf{vN})$  is an adjoint pair of functors).

- 4.6.** Let  $X$  and  $Y$  be topological vector spaces, and let  $\mathcal{U}$  be a neighborhood subbase at 0 in  $X$ . Show that a linear map  $\varphi: X \rightarrow Y$  is open if and only if for each  $U \in \mathcal{U}$   $\varphi(U)$  is a neighborhood of 0 in  $Y$ .
- 4.7.** Let  $X$  and  $Y$  be locally convex spaces, and let  $P$  and  $Q$  be fundamental families of seminorms on  $X$  and  $Y$ , respectively. Let  $\varphi: X \rightarrow Y$  be a continuous linear map. Show that
- (a)  $\varphi$  is topologically injective if and only if it is injective, and for each  $p \in P$  there exist  $c > 0$  and  $q_1, \dots, q_n \in Q$  such that  $\max_{1 \leq i \leq n} q_i(\varphi(x)) \geq cp(x)$  ( $x \in X$ ). Moreover, if  $X$  is Hausdorff, then the latter condition implies the injectivity of  $\varphi$ .
- (b)  $\varphi$  is open if and only if for each  $p \in P$  there exist  $C > 0$  and  $q_1, \dots, q_n \in Q$  such that for each  $y \in Y$  there exists  $x \in X$  satisfying  $\varphi(x) = y$  and  $p(x) \leq C \max_{1 \leq i \leq n} q_i(y)$ .