2.1. Show that any seminorm p on a vector space X equals the Minkowski functional of the open ball $U_p = \{x \in X : p(x) < 1\}$ and of the closed ball $\overline{U}_p = \{x \in X : p(x) \leqslant 1\}$.

2.2. Let X be a topological vector space, let $V \subset X$ be an absolutely convex neighborhood of 0, and let p_V denote the Minkowski functional of V. Show that $\operatorname{Int} V = \{x : p_V(x) < 1\}$ and $\overline{V} = \{x : p_V(x) \leqslant 1\}$. Deduce that $V \mapsto p_V$ is a 1-1-correspondence between the collection of all absolutely convex open neighborhoods of 0 and the collection of all continuous seminorms on X. Moreover, the inverse map is given by $p \mapsto U_p$.

2.3. Let p and q be seminorms on a vector space. Show that $p \leqslant q$ iff $U_q \subset U_p$, and that $p \prec q$ iff $U_q \prec U_p$.

Recall (see the lectures) that the space s of *rapidly decreasing sequences* is defined by

$$s = \Big\{ x = (x_n) \in \mathbb{K}^{\mathbb{N}} : \|x\|_k = \sup_{n \in \mathbb{N}} |x_n| n^k < \infty \ \forall k \in \mathbb{Z}_{\geqslant 0} \Big\}. \tag{1}$$

The topology on s is given by the seminorms $\|\cdot\|_k$ ($k \in \mathbb{Z}_{\geqslant 0}$). Similarly, one defines the space $s(\mathbb{Z})$ of rapidly decreasing sequences on $\mathbb{Z}$ (more exactly, we replace $\mathbb{N}$ by $\mathbb{Z}$ and n^k by $|n|^k$ in (1)).

2.4. Let $\lambda = (\lambda_n) \in \mathbb{K}^{\mathbb{N}}$. Consider the *diagonal operator*

$$M_\lambda \colon \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}, \quad (x_1, x_2, \ldots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \ldots).$$

(a) Show that M_λ is continuous.
(b) Find a condition on λ that is equivalent to $M_\lambda(s) \subset s$.
(c) Find a condition on λ that is necessary and sufficient for M_λ to be a continuous map of s to s.

2.5. Describe all continuous linear functionals on the spaces **(a)** $\mathbb{K}^{\mathbb{N}}$; **(b)** s.

Recall (see the lectures) that the *Schwartz space* $\mathscr{S}(\mathbb{R}^n)$ is defined by

$$\mathscr{S}(\mathbb{R}^n) = \Big\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty \ \forall \alpha, \beta \in \mathbb{Z}^n_{\geqslant 0} \Big\}.$$

The topology on $\mathscr{S}(\mathbb{R}^n)$ is given by the seminorms $\|\cdot\|_{\alpha,\beta}$ ($\alpha, \beta \in \mathbb{Z}^n_{\geqslant 0}$).

2.6. **(a)** Let $U \subset \mathbb{R}^n$ be an open set. Consider a differential operator

$$D = \sum_{|\alpha| \leqslant N} a_\alpha D^\alpha, \tag{2}$$

where $a_\alpha \in C^\infty(U)$. Show that D is a continuous operator on $C^\infty(U)$.
(b) Find a reasonable condition on $a_\alpha \in C^\infty(\mathbb{R}^n)$ that is sufficient for D to be a continuous operator on $\mathscr{S}(\mathbb{R}^n)$.
(c) Let us equip the space $\mathbb{K}[[x_1, \ldots, x_n]]$ of formal power series with the topology of convergence of each coefficient (in other words, we identify $\mathbb{K}[[x_1, \ldots, x_n]]$ with $\mathbb{K}^{\mathbb{Z}^n_{\geqslant 0}}$ equipped with the product topology). Show that for each $a_\alpha \in \mathbb{K}[[x_1, \ldots, x_n]]$ formula (2) defines a continuous operator on $\mathbb{C}[[x_1, \ldots, x_n]]$.
(d) Let $U \subset \mathbb{C}$ be an open set, and let $a_1, \ldots, a_N \in \mathscr{O}(U)$. Show that the differential operator

$$\sum_{k=0}^{N} a_k \frac{d^k}{dz^k}$$

is continuous on $\mathscr{O}(U)$.

2.7. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and let μ denote the normalized length measure on $\mathbb{T}$ ("normalized" means that the measure of $\mathbb{T}$ is 1). Show that the *Fourier transform*

$$\mathscr{F} : C^\infty(\mathbb{T}) \to s(\mathbb{Z}), \quad (\mathscr{F} f)(n) = \int_{\mathbb{T}} f(z) z^{-n}\, d\mu(z),$$

is a topological isomorphism of $C^\infty(\mathbb{T})$ onto $s(\mathbb{Z})$.

2.8. Let (X, μ) be a σ-finite measure space, and let $0 < p < 1$. We define $L^p(X, \mu)$ to be the space of (μ-equivalence classes of) measurable functions $f \colon X \to \mathbb{K}$ such that $|f|^p$ is integrable. Given $f \in L^p(X, \mu)$, let

$$|f|_p = \int_X |f(x)|^p \, d\mu(x).$$

(a) Show that $|\cdot|_p$ is an F-norm on $L^p(X, \mu)$. Thus $L^p(X, \mu)$ is a metrizable topological vector space.
(b) Show that the only continuous linear functional on $L^p[0, 1]$ is identically zero. As a corollary, $L^p[0, 1]$ is not locally convex.
(c) Can $L^p(X, \mu)$ be locally convex and infinite-dimensional?

2.9. Let (X, μ) be a finite measure space. We define $L^0(X, \mu)$ to be the space of (μ-equivalence classes of) all measurable functions $f \colon X \to \mathbb{K}$. Choose a bounded nondecreasing function $\varphi \colon [0, +\infty) \to [0, +\infty)$ satisfyiong the following conditions:

1) $\varphi(s + t) \leqslant \varphi(s) + \varphi(t) \quad (s, t \geqslant 0)$;
2) $\varphi(0) = 0$;
3) φ is a homeomorphism between suitable neighborhoods of 0.

For example, we can let $\varphi(t) = t/(1 + t)$ or $\varphi(t) = \min\{t, 1\}$. Given $f \in L^0(X, \mu)$, let

$$|f|_0 = \int_X \varphi(|f(x)|)\, d\mu(x).$$

(a) Show that $|\cdot|_0$ is an F-norm on $L^0(X, \mu)$. Thus $L^0(X, \mu)$ is a metrizable topological vector space.
(b) Show that a sequence in $L^0(X, \mu)$ converges iff it converges in measure.
(c) Show that the only continuous linear functional on $L^0[0, 1]$ is identically zero. As a corollary, $L^0[0, 1]$ is not locally convex.
(d) Can $L^0(X, \mu)$ be locally convex and infinite-dimensional?