7. INTEGRATION AND STOKES' THEOREM.

A manifold is called orientable if it has an oriented atlas.

Problem 1. Find formulations of the (a) Green's theorem, (b) divergence theorem (a.k.a. Ostrogradsky's or Gauss's theorem), (c) curl theorem (a.k.a. Kelvin–Stokes theorem) and derive them from the Stokes' theorem.

Problem 2. Let $S^n \subset \mathbb{R}^{n+1}$ be a unit sphere. Let $a = (a_1, \dots, a_{n+1}) \in S^n$ and $v_1 = (v_{11}, \dots, v_{1,n+1}), \dots, v_n = (v_{n1}, \dots, v_{n,n+1}) \in T_a S^n$. Define $\omega(a)(v_1, \dots, v_n) = \det \begin{pmatrix} a_1 & \dots & a_{n+1} \\ v_{11} & \dots & v_{1,n+1} \\ \dots & \dots & \dots \\ v_{n1} & \dots & v_{n,n+1} \end{pmatrix}$. Let $f : S^n \setminus \{(0, \dots, 0, 1)\} \to \mathbb{R}^n$ be a stereographic projection: $f(a)$ is the intersection point $\ell_a \cap \Pi$ where ℓ_a is the line joining a with $(0, \dots, 0, 1)$, and $\Pi \subset \mathbb{R}^{n+1}$ is the hyperplane given by the equation $x_{n+1} = 0$. (a) Compute explicitly the n-form $(f^{-1})^*\omega$ on $\mathbb{R}^n$. (b) Find the volume of the sphere $\operatorname{vol} S^n = \int_{S^n} \omega$ (for $n = 2, 3, 4$ if not for any n). Comment why it is reasonable to call this a volume (and ω, a volume form). (c) Let $m(R) = \int_{x_1^2 + \dots + x_{n+1}^2 \le R^2} dx_1 \wedge \dots \wedge dx_{n+1}$. Prove that $m'(1) = \int_{S^n} \omega$ (for any dimension n; try to give the proof not using explicit calculation of the right-hand side).