# 4. PARACOMPACTNESS AND PARTITION OF UNITY

Abstract. This problem set contains proofs of two important properties of smooth manifolds: paracompactness and the theorem about partition of unity. Proofs are partly written and partly given as exercises.

## 1. Paracompactness.

**Exercise 1.** Prove that if $M \subset \mathbb{R}^n$ is compact then for any $a \in \mathbb{R}^n \setminus M$ the distance $d(a, M) \stackrel{\text{def}}{=} \inf_{b \in M} |a - b| > 0$.

**Exercise 2.** Suppose a manifold $M$ has a countable pre-atlas. (a) Prove that $M$ possesses a countable pre-atlas $(U_\alpha, V_\alpha, x_\alpha)$ such that all closures $\overline{U_\alpha} \subset M$ and $\overline{V_\alpha} \subset \mathbb{R}^n$ are compact. (b) Prove that $M$ can be represented as a union $\bigcup_n M_n = M$ where $M_1 \subset M_2 \subset \dots \subset M$ are all open, and for any $n$ the closure $\overline{M_n} \subset M_{n+1}$ is compact.

**Hint.** In 2(a) first find a pre-atlas where all $V_\alpha$ are bounded (hence, $\overline{V_\alpha}$ are compact). Then consider sets $A_{\alpha,k} \stackrel{\text{def}}{=} \{a \in V_\alpha \mid d(a, \partial V_\alpha) > 1/k\}$ and $B_{\alpha,k} \stackrel{\text{def}}{=} x_\alpha^{-1}(A_k)$.

**Theorem 1.** *Let $M$ be a smooth manifold with a countable pre-atlas, and $\{P_\alpha\}$ be its cover by open sets: $M = \bigcup_\alpha P_\alpha$. Then there exist open sets $Q_{\alpha\beta}$ (where $\beta$ runs, for each $\alpha$, through an index set, depending on $\alpha$) such that*

1. *$Q_{\alpha\beta} \subset P_\alpha$ for all $\alpha$ and $\beta$.*
2. *$M = \bigcup_{\alpha,\beta} Q_{\alpha,\beta}$.*
3. *For every point $a \in M$ there exists an open set $U \ni a$ intersecting only finitely many sets $Q_{\alpha,\beta}$.*

In short: any open cover of a manifold admits a locally finite refinement.

*Proof.* Take the sequence of open sets $M_1 \subset M_2 \subset \dots \subset M$ from Exercise 2(b). For every $k$ the set $\overline{M_k} \setminus M_{k-1}$ is compact, so it can be covered by a finite collection of the sets $P_{\alpha_1^{(k)}}, \dots, P_{\alpha_{r_k}^{(k)}}$. Take $Q_{\alpha_i^{(k)},k} \stackrel{\text{def}}{=} (P_{\alpha_i^{(k)}} \setminus \overline{M_{k-2}}) \cap M_{k+1}$. All the sets $Q_{\alpha,k}$ are open, and $Q_{\alpha,k} \subset P_\alpha$ for all $\alpha$ and $k$. The sets $Q_{\alpha_1^{(k)},k}, \dots, Q_{\alpha_{r_k}^{(k)},k}$ cover $\overline{M_k} \setminus M_{k-1}$ and are contained in $M_{k+1} \setminus \overline{M_{k-1}}$. Hence, $\{Q_{\alpha,k}\}$ is locally finite. □

## 2. Partition of unity.

**Exercise 3.** (a) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = e^{-1/t^2}$ for $t \neq 0$ and $f(0) \stackrel{\text{def}}{=} 0$ has continuous derivative $f^{(n)}$ of any order $n$, and for $t \neq 0$ one has $f^{(n)}(t) = P_n(t) e^{-1/t^2}$ for some rational function $P_n$, and $f^{(n)}(0) = 0$ for all $n$. (b) Let $r > 0$. Prove that the function $g_r : \mathbb{R}^n \to \mathbb{R}$ defined as $g_r(x) \stackrel{\text{def}}{=} f(r^2 - |x|^2)$ for $|x| < r$ and $g_r(x) = 0$ for $|x| \geq r$ is smooth.

**Exercise 4.** (a) Let $M \subset \mathbb{R}^n$ be a compact. Prove that the function $\varphi_{M,r}(x) \stackrel{\text{def}}{=} \frac{\int_M g_r(y-x)\,dy}{\int_{|x| \leq r} g_r(y)\,dy}$ is smooth, where $g_r$ is the function of Exercise 3(b). Also prove that the support $\operatorname{supp} \varphi_{M,r} \stackrel{\text{def}}{=} \overline{\{x \in \mathbb{R}^n \mid \varphi_{M,r}(x) \neq 0\}}$ is compact. (b) Prove that $0 \leq \varphi_{M,r}(x) \leq 1$ for all $x \in \mathbb{R}^n$, and that $A_r \stackrel{\text{def}}{=} \varphi_{M,r}^{-1}(1) \subset M$. (c) Prove that $A_r \subset A_{r'}$ if $r > r'$. Prove that if $M$ is the closure of its own interior then $M = \bigcup_{r > 0} A_r$.

**Theorem 2** (partition of unity)**.** *Let manifold $M$ have a countable pre-atlas, and $\{P_\alpha\}$ be an open cover of $M$. Then there exist smooth functions $\varrho_\alpha : M \to \mathbb{R}$ (called a partition of unity subordinate to the cover $\{P_\alpha\}$) such that*

- *$\varrho_\alpha(x) \geq 0$ for all $x \in M$.*
- *$\operatorname{supp}(\varrho_\alpha) \subset P_\alpha$ for every $\alpha$.*
- *For every $a \in M$ there exists an open subset $U \ni a$ intersecting supports of only finitely many $\varrho_\alpha$.*
- *$\sum_\alpha \varrho_\alpha \equiv 1$.*

**Exercise 5.** (a) Prove that it suffices to prove a weaker statement, that $\sum_\alpha \varrho_\alpha(a) > 0$ for any $a \in M$. (b) Suppose that Theorem 2 is valid for a refinement $Q_{\alpha\beta}$ of the cover $P_\alpha$. Prove that it is valid for the cover $P_\alpha$ itself. (c) Suppose Theorem 2 is proved for any countable locally finite cover $\{P_\alpha\}$ such that all $P_\alpha$ are charts of a pre-atlas $(P_\alpha, V_\alpha, x_\alpha)$ where all the closures $\overline{P_\alpha} \subset M$ and $\overline{V_\alpha} \subset \mathbb{R}^n$ are compact. Prove then Theorem 2 for any cover.

*Proof of Theorem 2.* Let $P_1, P_2, \dots$ be a cover described in Exercise 5(c); prove now that the set $R_1 \stackrel{\text{def}}{=} x_1(P_1 \setminus \bigcup_{\beta \neq 1} P_\beta)$ is compact. If it is not the case, $R_1$ contains an infinite subset $A \subset R_1$ having no accumulation points. Once $x_1^{-1}(A) \subset \overline{P_1}$, and $\overline{P_1}$ is a compact, $x_1^{-1}(A)$ has an accumulation point $b \in M$. It means that $b \in P_i$ for some $i$. If $i = 1$ then $x_1(b) \in R_1$ is an accumulation point for $A$, contrary to its construction. So $i > 1$, and the intersection $P_i \cap x_1^{-1}(A)$ is nonempty. Therefore $A \cap x_1(P_i \cap P_1)$ is also nonempty — contrary to the assumption that $A \subset R_1$.

The set $R_1$ is compact, so similar to Exercise 1 one has $\varepsilon \stackrel{\text{def}}{=} d(R_1, \partial Q_1) > 0$. The set $\widetilde{R_1} = \bigcup_{x \in R_1} \overline{B}(x, \varepsilon/2) \subset Q_1$ has positive Lebesgue measure. At the same time $B(x, \varepsilon/2) \subset Q_1$ for all $x \in \widetilde{R_1}$. Take a function $\varrho_1 \stackrel{\text{def}}{=} \varphi_{R_1, \varepsilon/2} \in C^\infty(\mathbb{R}^n)$, obtained like in Exercise 4(a).

Denote now $W_1 = \{a \in M \mid \varrho_1(a) > 0\}$. By construction if $\varphi_1(q) = 0$ then $q \in x_1(P_1 \cap P_i)$ for some $i > 1$. Then the sets $W_1, P_2, P_3, \dots$ form a cover of $M$.

Define the functions $\varrho_n$, $n > 1$, in a similar fashion where $R_n$ is the set $x_n(P_n \setminus \bigcup_{i>n} P_\beta \setminus \bigcup_{j<n} W_j)$. The construction is valid because for any $n$ the sets $W_1, \dots, W_{n-1}, P_n, P_{n+1}$ is a cover of $M$. This implies the inequality $\sum_\alpha \varrho_\alpha > 0$. $\square$

**Exercise 6.** Find as many mistakes as possible in the preceding proof and correct them.

## 3. Algebra $C^\infty(M)$ of smooth functions.

**Exercise 7.** Let $M$ be a compact manifold. Show that if elements of an ideal $I \subseteq C^\infty(M)$ (not necessarily closed) have no common zeros then $I = C^\infty(M)$. Derive from this that any maximal ideal in $C^\infty(M)$ is $\mathcal{J}_a$ for some $a \in M$.

Let $M$ be not compact, and let $I \subset C^\infty(M)$ be the set of all functions $\varphi$ such that the support $\operatorname{supp} \varphi$ is compact.

**Exercise 8.** (a) Prove that $I$ is an ideal, $I \neq C^\infty(M)$ and the functions $\varphi \in I$ have no common zeros. Thus, Exercise 7 does not hold for non-compact manifolds. (b) Prove that $M$ has a maximal ideal which is not $\mathcal{J}_a$ for any $a$. Is this ideal closed?