1. TOPOLOGICAL SPACES AND SMOOTH MANIFOLDS

Problem 1. Let $M = S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ be covered by two charts: $U_1 = S^1 \setminus \{(0, 1)\}$ and $U_2 = S^1 \setminus \{(0, -1)\}$. Coordinate maps: $z_1 : U_1 \to V_1 = \mathbb{R}$ and $z_2 : U_2 \to V_2 = \mathbb{R}$ are projections from the points $(0, 1)$ and $(0, -1)$, resp., to the x-axis. Prove that the coordinate systems (U_1, V_1, z_1) and (U_2, V_2, z_2) define an 1-dimensional atlas on S^1 (after adding all "subsystems"). Prove that the topology on S^1 defined by this atlas conicides with the standard topology of the circle.

Problem 2. Let $\sim$ be an equivalence relation on a topological space X, and $Y \stackrel{\text{def}}{=} X/\sim$ is the set of equivalence classes. Call a subset $U \subset Y$ open if the union $\bigcup_{A \in U} A \subset X$ is open (in the topology of X). (a) Prove that the collection of open sets so defined makes Y a topological space; it is called the quotient topology. (b) Prove that the natural map $p : X \to X/\sim$ sending each point to its class of equivalence is continuous, and also that any map $f : X/\sim \to Y$ (where Y is another topological space) is continuous if and only if the composition $f \circ p : X \to Y$ is continuous. (c) Let $X = S^1$ and $a \sim b$ if and only if a and b are the two endpoints of a diameter of X. Prove that $X/\sim$ is homeomotphic to S^1. (d) Describe the quotient topology if $X = \mathbb{R}$ and $a \sim b$ if either $a = b = 0$ or $a \neq 0$ and $b \neq 0$.

Problem 3. The projective space $\mathbb{R}P^n$ is defined as the quotient $\mathbb{R}^{n+1} \setminus \{(0, \dots, 0)\}/\sim$ where $(u_0, \dots, u_n) \sim (v_0, \dots, v_n)$ if and only if there exists a $t \neq 0$ such that $v_0 = tu_0, \dots, v_n = tu_n$. An equivalence class of the point $(u_0, \dots, u_n)$ is denoted $[u_0 : \dots : u_n]$. The standard topology on $\mathbb{R}P^n$ is the quotient topology of Problem 2. An n-dimensional atlas on $\mathbb{R}P^n$ has $n + 1$ charts $U_0, \dots, U_n$ where $U_i \stackrel{\text{def}}{=} \{[u_0 : \dots : u_n] \mid u_i \neq 0\}$. The coordinate map $z_i : U_i \to V_i = \mathbb{R}^n$ is defined as $z_i([u_0 : \dots : u_n]) = (u_0/u_i, \dots, u_n/u_i)$ (the term u_i/u_i is skipped). (a) Prove that the coordinate systems (U_i, V_i, z_i), $i = 0, \dots, n$, define a n-dimensional smooth structure on $\mathbb{R}P^n$. (b) Prove that the topology on $\mathbb{R}P^n$ defined by this smooth structure is the same as the quotient topology. (c) The map $f : S^n \to \mathbb{R}P^n$ where $S^n \stackrel{\text{def}}{=} \{(u_0, \dots, u_n) \mid u_0^2 + \dots + u_n^2 = 1\} \subset \mathbb{R}^{n+1}$ is defined as $f(u_0, \dots, u_n) = [u_0 : \dots : u_n]$. Prove that this map is smooth. (d) Prove that for any point $a \in \mathbb{R}P^n$ there exists an open set $U \ni a$ such that $f^{-1}(U) \subset S^m$ is homeomorphic to a disjoint union of two copies of U.

Recall that a topological space X is called connected if an open subset $U \subset X$ such that $X \setminus U$ is also open is empty or equal to X.

Problem 4. (a) Prove that the topological space $[0, 1]$ is connected. (b) Suppose that the topological space X has the following property (called arcwise connectedness): if $a, b \in X$ then there exists a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = a$, $\gamma(1) = b$ (that is, a curve joining a and b). Prove that X is connected. The converse is false: invent a counterexample or look for it in any topology textbook. (c) Let M be a smooth manifold and $a \in M$. Prove that the set U_a of all points $b \in M$ such that there exists a curve in M joining a with b is open. (d) Prove that the set $M \setminus U_a$ is also open. Derive from it that a smooth manifold is connected if and only if it is arcwise connected.

Problem 5*. (a) Prove that if a 1-dimensional manifold is homeomorphic to S^1 then it is diffeomorphic to S^1. (b) The same question for $\mathbb{R}^1$. (c*) Prove that a connected smooth 1-dimensional manifold is diffeomorphic to either $\mathbb{R}$ or S^1.