LECTURE 9

ABSTRACT. Poincaré's lemma and degree of a smooth map.

A k-form ω is called *closed* if $d\omega = 0$. The form is called *exact* if there exists a $(k-1)$-form ν such that $\omega = d\nu$.

Example 1. A 0-form (i.e. a smooth function) f is closed if and only if it is locally constant, that is, constant on every connected component of the manifold M. A nonzero 0-form cannot be exact (because there are no -1-forms).

Example 2. Any n-form on a n-manifold M is closed (because there are no $(n+1)$-forms on M).

An exact form is always closed because $d^2 = 0$. The converse is not always true: let ω be a n-form on a compact n-manifold M; it is closed by Example 2. If $\omega = d\nu$ then $\int_M \omega = 0$ by Stokes' theorem. So if $\int_M \omega \neq 0$ (which is true, for example, for a volume form on S^n) then ω is not exact.

Theorem 1 (Poincaré's lemma)**.** *Let ω be a closed k-form on $\mathbb{R}^n$ with $1 \le k \le n$. Then ω is exact.*

Proof. Induction by n. For $n = 1$ one has $k = 1$ and $\omega = f(x)\,dx$ for some smooth function $f : \mathbb{R}^1 \to \mathbb{R}$. This form is closed by Example 2; it is also exact: $\omega = dg$ where $g(x) \stackrel{\text{def}}{=} \int_0^x f(t)\,dt$.

Let now the theorem be proved for $\mathbb{R}^{n-1}$. Represent $\mathbb{R}^n = \mathbb{R}^1 \times \mathbb{R}^{n-1}$ and denote the coordinate in $\mathbb{R}^1$ by t and the coordinates in $\mathbb{R}^{n-1}$, by $x = (x_1, \dots, x_{n-1})$. Then $\omega = \omega_1(t,x) + dt \wedge \omega_2(t,x)$ where the k-form ω_1 and the $(k-1)$-form ω_2 do not contain dt.

One has $d\omega = d_x\omega_1 + dt \wedge \frac{\partial \omega_1}{\partial t} - dt \wedge d_x\omega_2$ where d_x is the exterior derivative with respect to the x variables, that is, the exterior derivative on the manifold (a hyperplane) $M_t = \{t\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$; once the forms ω_1 and ω_2 do not contain dt, they can be considered as forms on M_t. (The partial derivative $\frac{\partial}{\partial t}$ is meant as usual: ω_1 is a map from $\mathbb{R}^1$ to the vector space $\Omega^k(\mathbb{R}^{n-1})$ of the k-forms in $\mathbb{R}^{n-1}$, so one can take a derivative.) It follows from $d\omega = 0$ that $d_x\omega_1 = 0$ and $\frac{\partial \omega_1}{\partial t} = d_x\omega_2$.

Let $\nu = \nu_1 + dt \wedge \nu_2$ be a $(k-1)$-form where ν_1 and ν_2 do not contain dt. The equation $d\nu = \omega$ is equivalent to $d_x\nu_1 = \omega_1$ and $d_x\nu_2 = \frac{\partial \nu_1}{\partial t} - \omega_2$. Since $d_x\omega_1 = 0$ the equation $d_x\nu_1 = \omega_1$ is solvable on any hyperplane $M_t = \mathbb{R}^{n-1}$ by the induction hypothesis; take a solution smooth in t. Now $d_x\left(\frac{\partial \nu_1}{\partial t} - \omega_2\right) = \frac{\partial}{\partial t} d_x\nu_1 - d_x\omega_2 = \frac{\partial \omega_1}{\partial t} - d_x\omega_2 = 0$, hence the equation $d_x\nu_2 = \frac{\partial \nu_1}{\partial t} - \omega_2$ is also solvable by the induction hypothesis. □

The situation is somewhat different if we limit our considerations to the forms with the compact support. Consider only the top degree forms:

Proposition 1. *Let ω be a n-form on $\mathbb{R}^n$ with $\operatorname{supp}\omega \subset (0,1)^n$. The equality $\int_{\mathbb{R}^n} \omega = 0$ takes place if and only if there exists a $(n-1)$-form ν such that $d\nu = \omega$ and $\operatorname{supp}\nu \subset (0,1)^n$.*

Proof. Let the form ν exist; then by Stokes' theorem $\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} d\nu = 0$.

The converse statement for $n = 1$ is proved similar to the induction base in the Poincaré's lemma: if $\omega = f(t)\,dt$ with $\operatorname{supp} f \subset (0,1)$ and $\int_0^1 f(t)\,dt = 0$ then $\omega = dg$ where $g(x) \stackrel{\text{def}}{=} \int_0^x f(t)\,dt$; if $x \ge 1$ then $g(x) = \int_0^1 f(t)\,dt = 0$, so $\operatorname{supp} g \subset (0,1)$.

Let now $n > 1$ and $\omega = f(x,t)\,dt \wedge dx$ where $x = (x_1, \dots, x_{n-1})$ and dx stands for $dx_1 \wedge \dots \wedge dx_{n-1}$; suppose that $\int_0^1 \dots \int_0^1 f(x,t)\,dt dx = 0$. Take $\mu \stackrel{\text{def}}{=} \left(\int_0^1 f(x,t)\,dt\right) dx$. Apparently, $\operatorname{supp}\omega \subset (0,1)^{n-1}$ and $\int_{\mathbb{R}^{n-1}} \mu = 0$. Induction by n shows that there exists a $(n-2)$-form ψ on $\mathbb{R}^{n-1}$ such that $\mu = d\psi$ and $\operatorname{supp}\psi \subset (0,1)^{n-1}$.

Let φ be the function of 1 variable such that $\operatorname{supp}\varphi \subset (0,1)$ and $\int_0^1 \varphi(\tau)\,d\tau = 1$. Take now Consider now the form

$$\nu = \left(\int_0^t \left(f(x,s) - \varphi(s) \int_0^1 f(x,\tau)\,d\tau\right) ds\right) dx - \varphi(t) dt \wedge \psi.$$

It is easy to check that $\operatorname{supp}\nu \subset (0,1)^n$; one has also $d\nu = f(x,t)\,dt \wedge dx - \varphi'(t)\left(\int_0^1 f(x,s)\,ds\right) dt \wedge dx + \varphi'(t) dt \wedge d\psi = f(x,t)\,dt \wedge dx = \omega$ because $d\psi = \mu = \left(\int_0^1 f(x,s)\,ds\right) dx$. □

Let $f : M \to N$ be a smooth map between oriented manifolds of the same dimension n. For $p \in M$ let $U \subset M$ be a chart containing p and $W \subset N$, a chart containing $f(p)$; the corresponding coordinates are x and y. Let the point p be regular, that is, $f'(p) : T_pM \to T_{f(p)}N$ be nondegenerate, so $\det(y \circ f \circ x^{-1})'(x(p)) \neq 0$. If $\tilde{U}$, $\tilde{W}$ are other charts containing p and $f(p)$, and $\tilde{x}$, $\tilde{y}$ are the corresponding coordinates with the transition maps φ and ψ, respectively, then $\det(\tilde{y} \circ f \circ \tilde{x}^{-1})'(\tilde{x}(p)) = \det(\varphi \circ y \circ f \circ x^{-1} \circ \psi)'(\tilde{x}(p)) = \det\varphi' \det\psi' \det(y \circ f \circ x^{-1})'(x(p))$.

The atlases in M and N are oriented, so $\det\varphi', \det\psi' > 0$, and $\det(\tilde{y}\circ f\circ\tilde{x}^{-1})'(\tilde{x}(p))$ and $\det(y\circ f\circ x^{-1})'(x(p))$ have the same sign. This sign is called the sign of the regular point p with respect to the map f and is denoted $\operatorname{sign}(p, f)$.

Let M be compact. Take a point $a \in N$ and let $U \subset N$ be an open subset diffeomorphic to $(0,1)^n$ (that is, to $\mathbb{R}^n$) such that $a \in U$. Consider a n-form ω on N with the support $\operatorname{supp}\omega \subset U$ and $\int_N \omega = 1$. Denote $\deg(f, a, \omega) \stackrel{\text{def}}{=} \int_M f^*\omega$.

Theorem 2. *The number* $\deg(f, a, \omega) = \deg(f, a)$ *does not depend on the choice of a particular form* ω*. The map* $a \mapsto \deg(f, a)$ *is locally constant; hence if* N *is connected then* $\deg(f, a) = \deg(f)$ *does not depend on* a *either. If* a *is a regular value of* f *(that is, any point* $p \in f^{-1}(a)$ *is regular with respect to* f*) then* $\deg(f, a) = \sum_{p\in f^{-1}(a)} \operatorname{sign}(p, f) \in \mathbb{Z}$.

Proof. Let ω_1 be another form on N with the required properties. Then $\int_N(\omega - \omega_1) = 0$. The forms ω and ω_1 are defined on $U = \mathbb{R}^n$; thus by Proposition 1 there exists a $(n-1)$-form ν such that $\omega = \omega_1 + d\nu$ and $\operatorname{supp}\nu$ is compact. This implies $\int_M f^*\omega = \int_M f^*\omega_1 + \int_M f^*d\nu = \int_M f^*\omega_1 + \int_M df^*\nu = \int_M f^*\omega_1$ by the Stokes' theorem.

If a point a_1 is sufficiently close to a then one can choose a common U such that $a, a_1 \in U$ and one ω with $\operatorname{supp}\omega \subset U$. Then $\deg(f, a) = \deg(f, a, \omega) = \deg(f, a_1, \omega) = \deg(f, a_1)$, that is, the function $\deg(f, a)$ is locally constant on N. If N is connected then $\deg(f, a)$ is constant.

Let now a be a regular value of f. By the inverse function theorem, for every $b \in f^{-1}(a)$ there exists a neighbourhood $U_b \ni b$ mapped diffeomorphically by f to a neighbourhood of a and therefore containing no points of $f^{-1}(a)$ other than b. Thus $f^{-1}(a) \subset N$ is discrete; since N is compact, it means that $f^{-1}(a) = \{b_1, \dots, b_N\}$ is finite. Denote $U_i \stackrel{\text{def}}{=} U_{b_i}$. The set $W \stackrel{\text{def}}{=} \bigcup_i f(U_i) \ni a$ is open; without loss of generality it is a chart.

Let $\operatorname{supp}\omega \subset W$, then $f^*\omega = \sum_{i=1}^N \left(f|_{U_i}\right)^*\omega$, and $\deg(f) = \int_M f^*\omega = \sum_{i=1}^N \int_M \left(f|_{U_i}\right)^*\omega = \sum_{i=1}^N \int_{U_i} f^*\omega = \sum_{i=1}^N \operatorname{sign}(b_i, f) \int_N \omega = \sum_{i=1}^N \operatorname{sign}(b_i, f)$. ☐

Corollary 1. *If* M *is compact and oriented and* N *is connected and oriented then the sum* $\sum_{b\in f^{-1}(a)} \operatorname{sign}(b, f)$ *is the same for any regular value* a *of* f.

If $f^{-1}(a) = \varnothing$ then a is a regular value with $\deg(f, a) = 0$. So, Corollary 1 implies

Corollary 2. *If* M *is compact and oriented and* N *is connected and oriented and* $\deg(f) \neq 0$ *then* $f(M) = N$*. In particular,* $\deg(f) \neq 0$ *is possible only if* N *is compact.*