LECTURE 8

ABSTRACT. Lie derivative and Cartan's formula.

Let X be a vector field on a manifold M, and $\omega \in \Omega^k(M)$. Take $a \in M$; $U \ni a$ be a neighbourhood of a, and let $\Phi_X : U \times [-\varepsilon, \varepsilon] \to M$ be a local phase flow of the field X: $\Phi_X(b,t) \stackrel{\text{def}}{=} \gamma(t)$ where γ is the phase curve of the field X with $\gamma(0) = b$. For a fixed t the map $\Phi_t^X \stackrel{\text{def}}{=} \Phi_X|_{U\times\{t\}} : U \to M$ is smooth, so one can consider the k-form $\omega_t \stackrel{\text{def}}{=} (\Phi_t^X)^*\omega$. The k-form $\mathcal{L}_X\omega \stackrel{\text{def}}{=} \frac{d\omega_t}{dt}\big|_{t=0}$ is called the Lie derivative of the form ω with respect to the field X. For $k = 0$ then the operator $\mathcal{L}_X : \Omega^0(M) = C^\infty(M) \to C^\infty(M)$ is the action of the vector field X on smooth functions defined earlier.

Denote by $\iota_X\omega$ the $(k-1)$-form defined at a point $a \in M$ by the formula $(\iota_X\omega)(a)(v_1,\dots,v_{k-1}) = \omega(a)(X(a), v_1,\dots,v_{k-1})$ for all $v_1,\dots,v_{k-1} \in T_aM$. If $k = 0$ then take $\iota_X\omega = 0$ by definition.

A linear operator A on the super-commutative algebra $\Omega(M)$ of all differential forms (with the wedge product as multiplication) is called a derivation if $A(\omega_1 \wedge \omega_2) = (A\omega_1) \wedge \omega_2 + \omega_1 \wedge (A\omega_2)$ for all $\omega_1, \omega_2 \in \Omega(M)$; the operator is called a super-derivation if $A(\omega_1 \wedge \omega_2) = (A\omega_1) \wedge \omega_2 + (-1)^k\omega_1 \wedge (A\omega_2)$ for $\omega_1 \in \Omega^k(M)$ and any ω_2.

Theorem 1. (1) *The operator $\mathcal{L}_X$ is a derivation, the operators ι_X and d are super-derivations.*
(2) *$\iota_X\iota_Y = -\iota_Y\iota_X$ for any vector fields X and Y.*
(3) *$\mathcal{L}_X d = d\mathcal{L}_X$ for any vector field X.*
(4) *$\mathcal{L}_X\mathcal{L}_Y = \mathcal{L}_Y\mathcal{L}_X + \mathcal{L}_{[X,Y]}$ for all X and Y.*
(5) *(Cartan's formula) $\mathcal{L}_X = \iota_X d + d\iota_X$ for any X.*
(6) *$\iota_X\mathcal{L}_Y = \mathcal{L}_Y\iota_X + \iota_{[X,Y]}$ for all X and Y.*

Proof. 1. By naturality of the wedge product $(\Phi_t^X)^*(\omega_1 \wedge \omega_2) = ((\Phi_t^X)^*\omega_1) \wedge ((\Phi_t^X)^*\omega_2)$, which implies that $\mathcal{L}_X$ is a derivation. ι_X is obviously a super-derivation, and the fact that d is a super-derivation was proved earlier.

2 follows from the fact that $\omega(a)$ is a skew-symmetric k-linear form on T_aM.

3. By naturality of the exterior derivative one has $d(\Phi_t^X)^*\omega = (\Phi_t^X)^*d\omega$, hence $\mathcal{L}_X d\omega = d\mathcal{L}_X\omega$.

4. By Statement 3 the commutator $[\mathcal{L}_X, \mathcal{L}_Y] \stackrel{\text{def}}{=} \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$ commutes with d and maps $\Omega^k(M)$ to $\Omega^k(M)$ (preserves the degree of a form). By Statement 1 the $\mathcal{L}_X$ and $\mathcal{L}_Y$ are derivations, and therefore $[\mathcal{L}_X, \mathcal{L}_Y]$ is a derivation too. Locally any form ω is a sum of expressions like $f \wedge dx_1 \wedge \dots \wedge dx_k$ where $f, x_1, \dots, x_k$ are functions (0-forms), so it will suffice to prove the equality $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ on functions. This was done earlier.

5. Denote the right-hand side of the equality by $\{\iota_X, d\}$ and call it a super-commutator. The operator $\{\iota_X, d\}$ preserves the degree of a form and commutes with d: $d\{\iota_X, d\} = d\iota_X d = \{\iota_X, d\}d$ because $d^2 = 0$. Once ι_X and d are proved to be super-derivations, $\{\iota_X, d\}$ is a derivation (the proof of this is an easy exercise similar to the statement that the commutator of two derivations is a derivation). Therefore, similar to Statement 4, it suffices to prove the Cartan's formula for 0-forms, that is, functions. Since the equality deals with the forms and vector fields locally (to compute all operators at a point $a \in M$ it is enough to know the fields and the forms in any neighbourhood of a), one can suppose without loss of generality that M is an open subset of $\mathbb{R}^n$. Then $\omega = f(x)$, the vector field $X = \sum_{i=1}^n A_i(x)\frac{\partial}{\partial x_i}$, so that $\iota_X f = 0 \Longrightarrow d\iota_X f = 0$ and $\iota_X df = \sum_{i=1}^n A_i(x)\frac{\partial f}{\partial x_i} = \mathcal{L}_X f$.

6. One has $\{[\iota_X, \mathcal{L}_Y], d\} = d\iota_X\mathcal{L}_Y - d\mathcal{L}_Y\iota_X + \iota_X\mathcal{L}_Y d - \mathcal{L}_Y\iota_X d = d\iota_X\mathcal{L}_Y - \mathcal{L}_Y d\iota_X + \iota_X d\mathcal{L}_Y - \mathcal{L}_Y\iota_X d$ (by Statement 3) $= [\{\iota_X, d\}, \mathcal{L}_Y] = [\mathcal{L}_X, \mathcal{L}_Y]$ (by Statement 5) $= \mathcal{L}_{[X,Y]}$ (by Statement 4) $= \{\iota_{[X,Y]}, d\}$ (by Statement 5). The operator ι_X is a super-derivation, $\mathcal{L}_Y$, a derivation, so $[\iota_X, \mathcal{L}_Y]$ is a super-derivation. Thus it suffices to check the equality in question on 0-forms (functions) where it looks as $0 = 0$. □