LECTURE 7

Abstract. Integration of differential forms. Stokes' theorem.

1. **Integration of forms.** For a differential form ω its support $\operatorname{supp}\omega$ is defined as the closure of the set $\{a \in M \mid \omega(a) \neq 0\} \subset M$. From now on let M be an oriented n-dimensional manifold and ω, a n-form (the same n!) with compact support on it.

Let first $M = U \subset \mathbb{R}^n$ be an open subset. Then the form ω looks as $\omega(x) = f(x)\,dx_1 \wedge \dots \wedge dx_n$ where $\operatorname{supp} f \subset U$ is compact. The integral $\int_U \omega$ is defined as the Riemann integral $\int_U f(x)\,dx_1 \dots dx_n$ (f is smooth, hence continuous, and therefore Riemann integrable).

Lemma 1. *Let $U_1, U_2 \subset \mathbb{R}^n$ be open subsets, and $h : U_1 \to U_2$ be a diffeomorphism such that $\det h'(x) > 0$ for any $x \in U_1$. Let $\operatorname{supp}\omega \subset U_2$ be compact. Then $\int_{U_2} \omega = \int_{U_1} h^*\omega$.*

Proof. Let $h(x) = (h_1(x_1, \dots, x_n), \dots, h_n(x_1, \dots, x_n))$; then

$$\begin{aligned}
h^*\omega(x) &= f(h(x))dh^*x_1 \wedge \dots \wedge dh^*x_n = f(h(x))dh_1(x) \wedge \dots \wedge dh_n(x)\\
&= \sum_{i_1,\dots,i_n=1}^{n} f(h(x))\frac{\partial h_1}{\partial x_{i_1}} \dots \frac{\partial h_n}{\partial x_{i_n}}\, dx_{i_1} \wedge \dots \wedge dx_{i_n} = \sum_{\sigma \in S_n} (-1)^{\text{parity of }\sigma} \frac{\partial h_1}{\partial x_{i_1}} \dots \frac{\partial h_n}{\partial x_{i_n}}\, dx_1 \wedge \dots \wedge dx_n\\
&= \det h'(x) f(h(x))\, dx_1 \wedge \dots \wedge dx_n.
\end{aligned}$$

Now the lemma follows from the change of variables formula for the n-dimensional integral (the absolute value sign in this formula is unnecessary here because $\det h'(x) > 0$). □

Let now M be any manifold with an oriented atlas $\{(U_\alpha.V_\alpha, x_\alpha)\}$, and let $\{\varrho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$.

Definition 1. $\int_M \omega = \sum_\alpha \int_{V_\alpha} (x_\alpha^{-1})^*(\varrho_\alpha\omega)$.

Proposition 1. *Definition 1 is sound: if $\{(\tilde{U}_\alpha.\tilde{V}_\alpha, \tilde{x}_\alpha)\}$ is another atlas on M with the same orientation, and $\tilde{\varrho}_\alpha$ is a partition of unity subordinate to $\{\tilde{U}_\alpha\}$ then $\int_M \omega$ defined by means of this partition is equal to that of Definition 1.*

Proof. Let $a \in U_\alpha \cap \tilde{U}_\beta$, $V_{\alpha\beta} = x_\alpha(U_\alpha \cap \tilde{U}_\beta) \subset V_\alpha$, $V_{\beta\alpha} = \tilde{x}_\beta(U_\alpha \cap \tilde{U}_\beta) \subset \tilde{V}_\beta$. The transition map $\varphi_{\beta\alpha} = x_\alpha \tilde{x}_\beta^{-1} : V_{\beta\alpha} \to V_{\alpha\beta}$ is a diffeomorphism; once the two atlases have the same orientation, $\det \varphi'_{\beta\alpha}(u) > 0$ for any u.

Now one has

$$\begin{aligned}
\textstyle\sum_\alpha \quad \int_{V_\alpha} (x_\alpha^{-1})^*(\varrho_\alpha\omega) &= \sum_{\alpha,\beta} \int_{V_\alpha} (x_\alpha^{-1})^*(\varrho_\alpha\tilde{\varrho}_\beta\omega) && \text{because } \textstyle\sum_\beta \tilde{\varrho}_\beta \equiv 1\\
&= \sum_{\alpha,\beta} \int_{V_{\alpha\beta}} (x_\alpha^{-1})^*(\tilde{\varrho}_\beta\varrho_\alpha\omega) && \text{because } \operatorname{supp}\varrho_\alpha \cap \operatorname{supp}\tilde{\varrho}_\beta \subset U_\alpha \cap \tilde{U}_\beta\\
&= \sum_{\alpha,\beta} \int_{V_{\beta\alpha}} \varphi_{\beta\alpha}^*(x_\alpha^{-1})^*(\tilde{\varrho}_\beta\varrho_\alpha\omega) && \text{by Lemma 1}\\
&= \sum_{\alpha,\beta} \int_{V_{\beta\alpha}} \varrho_\alpha(\tilde{x}_\beta^{-1})^*(\tilde{\varrho}_\beta\omega) = \textstyle\sum_\beta \int_{V_\beta} (\tilde{x}_\beta^{-1})^*(\tilde{\varrho}_\beta\omega) && \text{because } \textstyle\sum_\alpha \varrho_\alpha \equiv 1.
\end{aligned}$$

□

Example 1. Let $S^1 \subset \mathbb{R}^2$ be a unit circle centered at the origin. The tangent space T_aS^1 is naturally identified with the line $\{v \in \mathbb{R}^2 \mid (a,v) = 0\}$ where $(\cdot,\cdot)$ is the standard scalar product. Let $\nu_a : T_aS^1 \to \mathbb{R}$ be a linear functional such that for any $v \in T_aS^1$ one has $\nu_a(v) = |v|$ if v is directed counterclockwise from the vector a, and $\nu_a(v) = -|v|$ if v is directed clockwise. Thus $a \mapsto \nu_a$ is a differential 1-form on S^1.

Consider at S^1 an atlas $\{(U_1, (-\pi,\pi), \varphi), (U_2, (-\pi,\pi), \tilde{\varphi})\}$ described at Example 5 of Lecture 6: $U_1 = S^1 \setminus \{a\}$, $U_2 = S^1 \setminus \{b\}$ where $a, b \in S^1$ are two opposite points, φ and $\tilde{\varphi}$ are polar angles measured counterclockwise with $\varphi(b) = 0$ and $\tilde{\varphi}(a) = 0$, respectively. The transition maps are $\psi(u) = u - \pi$ in the lower arc ab and $\psi(u) = u + \pi$ in the upper arc. The 1-form described above is $d\varphi = d\tilde{\varphi}$.

Let $\varrho_1, \varrho_2 = 1 - \varrho_1$ be a partition of unity subordinate to $\{U_1, U_2\}$. Thus, $\operatorname{supp}\varrho_1 \subset U_1$, so $\varrho_1(x) = 0 \Longrightarrow \varrho_2(x) = 1$ in a neighbourhood of a, and vice versa in a neighbourhood of b. Hence $\varphi^*(\varrho_1 d\varphi) = f(t)\,dx$ where $f : (-\pi,\pi)$ is a smooth function such that $f(t) \equiv 1$ in a neighbourhood of $t = 0$ and $f(t) \equiv 0$ in neighbourhoods of $t = \pm\pi$. From the formulas for the transition maps it follows then that $\tilde{\varphi}^*((1 - \varrho_1)d\tilde{\varphi}) = g(t)\,dt$ where $g(t) = 1 - f(t + \pi)$

for $t \in (-\pi, 0)$ and $g(t) = 1 - f(t - \pi)$ for $t \in (0, \pi)$ (in particular, $g(t) \equiv 1$ in a neighbourhood of $t = 0$). Thus, $\int_{S^1} \nu = \int_{-\pi}^{\pi} (f(t) + g(t))\, dt = 2\pi$.

2. **Manifolds with boundary and Stokes' theorem.** An *atlas with boundary* in the set M is the set of triples $\{(U_\alpha.V_\alpha, x_\alpha)\}$ having all the properties of an atlas on a manifold, with one exception: for some systems of coordinates $V_\alpha \subset \mathbb{R}^n$ is an open set (like for a manifold; these are called internal systems), while for other systems of coordinates $V_\alpha = \tilde{V}_\alpha \cap \mathbb{R}^n_-$ where $\tilde{V}_\alpha \subset \mathbb{R}^n$ is an open subset and $\mathbb{R}^n_- \stackrel{\text{def}}{=} \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \le 0\}$; these are called boundary systems. Transition maps $\varphi_{\alpha\beta}$ now are either smooth maps defined in open subsets $V_{\alpha\beta} \subset \mathbb{R}^n$ or restrictions of smooth maps defined in open subsets $\tilde{V}_{\alpha\beta} \subset \mathbb{R}^n$ to the intersections $V_{\alpha\beta} = \tilde{V}_{\alpha\beta} \cap \mathbb{R}^n_-$.

Definition. A point $a \in M$ is called a point of the boundary if $a \in U_\alpha$ for some boundary system of coordinates $(U_\alpha, V_\alpha, x_\alpha)$ and $x_\alpha(a) \in \mathbb{R}^n_0 \stackrel{\text{def}}{=} \{(0, x_2, \dots, x_n) \mid x_2, \dots, x_n \in \mathbb{R}\}$. The boundary (the set of all points of the boundary) is denoted ∂M.

Lemma 2. *If $a \in \partial M$ and $a \in U_\alpha$ for some system of coordinates $(U_\alpha, V_\alpha, x_\alpha)$ then this system of coordinates is boundary and $x_\alpha(a) \in \mathbb{R}^n_0$.*

Proof. Suppose $x_\alpha(a)$ is an internal point of V_α (that is, $(U_\alpha, V_\alpha, x_\alpha)$ is an internal system of coordinates, or it is a boundary system of coordinates but $x_\alpha(a) \notin \mathbb{R}^n_0$). Since $a \in \partial M$, there exists a boundary system of coordinates $(U_\beta, V_\beta, x_\beta)$ such that $a \in U_\beta$ and $x_\beta(a) \in \mathbb{R}^n_0$. The transition map $\varphi_{\beta\alpha} : V_{\beta\alpha} \to V_{\alpha\beta}$ is a restriction of a diffeomorphism $\varphi_{\beta\alpha} : \tilde{V}_{\beta\alpha} \to W \subset \mathbb{R}^n$ defined in an open subset $\tilde{V}_{\beta\alpha} \subset \mathbb{R}^n$ to the intersection $V_{\beta\alpha} = \tilde{V}_{\beta\alpha} \cap \mathbb{R}^n_-$. By assumption we have $\psi_{\beta\alpha}(x_\beta(a)) = x_\alpha(a) \notin \mathbb{R}^n_0$; hence there exists $\varepsilon > 0$ such that the ball $B_\varepsilon(x_\alpha(a)) = \{y \mid |y - x_\alpha(a)| < \varepsilon\}$ is a subset of $V_{\alpha\beta}$. The map $\psi_{\beta\alpha}$ is continuous, so there exists a $\delta > 0$ such that $\psi_{\beta\alpha}(B_\delta(x_\beta(a))) \subset B_\varepsilon(x_\alpha(a)) \subset V_{\alpha\beta}$. Thus $\psi_{\alpha\beta}(V_{\alpha\beta}) \supset B_\delta(x_\beta(a))$. Since $x_\beta(a) \in \mathbb{R}^n_0$, the intersection $B_\delta(x_\beta(a)) \cap \mathbb{R}^n_+ \ne \varnothing$, so $\psi_{\alpha\beta}(V_{\alpha\beta}) \cap \mathbb{R}^n_+ \ne \varnothing$ contrary to $\psi_{\alpha\beta}(V_{\alpha\beta}) = V_{\beta\alpha} \subset \mathbb{R}^n_-$. □

Denote by $p : \mathbb{R}^n_0 \to \mathbb{R}^{n-1}$ the natural identification (deletion of the leading zero). Let $(U_\alpha, V_\alpha, x_\alpha)$ be a boundary coordinate system on M. Take $\tilde{U}_\alpha \stackrel{\text{def}}{=} U_\alpha \cap \partial M$, $\tilde{V}_\alpha \stackrel{\text{def}}{=} p(V_\alpha \cap \mathbb{R}^n_0) \subset \mathbb{R}^{n-1}$ and $\tilde{x}_\alpha \stackrel{\text{def}}{=} p \circ x_\alpha|_{\partial M}$.

Theorem 1. *$\{(\tilde{U}_\alpha, \tilde{V}_\alpha, \tilde{x}_\alpha)\}$ is an $(n-1)$-dimensional atlas for ∂M. If $\{(U_\alpha, V_\alpha, x_\alpha)\}$ is oriented then this atlas is oriented, too.*

Proof. Proof of the properties of an atlas for $\{(\tilde{U}_\alpha, \tilde{V}_\alpha, \tilde{x}_\alpha)\}$ is a routine check. The only statement worth proving is orientability.

Let $\psi_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$ be a transition map between two boundary coordinate systems on M; explicitly $\psi_{\alpha\beta}(u_1, \dots, u_n) \stackrel{\text{def}}{=} (f_1(u_1, \dots, u_n), \dots, f_n(u_1, \dots, u_n))$. Here $u_1 \le 0$, $f_1(u_1, \dots, u_n) \le 0$, and by Lemma 2 $f_1(0, u_2, \dots, u_n) \equiv 0$. Hence $\psi'_{\alpha\beta}(0, u_2, \dots, u_n) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1}(0, u_2, \dots, u_n) & 0 & \dots & 0 \\ \frac{\partial f_2}{\partial u_1}(0, u_2, \dots, u_n) & & & \\ \vdots & & \tilde{\psi}'_{\alpha\beta}(u_2, \dots, u_n) & \\ \frac{\partial f_n}{\partial u_1}(0, u_2, \dots, u_n) & & & \end{pmatrix}$ where $\tilde{\psi}_{\alpha\beta}$ is the transition map for the atlas on ∂M. So $\det \psi'_{\alpha\beta}(0, u_2, \dots, u_n) = \frac{\partial f_1}{\partial u_1}(0, u_2, \dots, u_n) \det \tilde{\psi}'_{\alpha\beta}(u_2, \dots, u_n) > 0$. Since $f_1(u_1, \dots, u_n) \le 0$ for all $u_1 \le 0$, one has $\frac{\partial f_1}{\partial u_1}(0, u_2, \dots, u_n) > 0$, and therefore $\det \tilde{\psi}'_{\alpha\beta}(u_2, \dots, u_n) > 0$. □

Theorem 2 (Stokes)**.** *Let M be an oriented manifold with boundary, and $\iota : \partial M \to M$ is the tautological embedding ($\iota(a) = a$ for any $a \in \partial M \subset M$). Let ω be a $(n-1)$-form with compact support on M. Then $\int_M d\omega = \int_{\partial M} \iota^* \omega$.*

Proof. Let first $M = \mathbb{R}^n_- = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \le 0\}$, so $\partial M = \mathbb{R}^{n-1}$. Then $\omega = f_1(x)\, dx_2 \wedge \dots \wedge dx_n + f_2(x)\, dx_1 \wedge dx_3 \wedge \dots \wedge dx_n + \dots + f_n(x)\, dx_1 \wedge \dots \wedge dx_{n-1}$, $\iota^* \omega = f_1(0, x_2, \dots, x_n)\, dx_2 \wedge \dots \wedge dx_n$ and $d\omega = (\frac{\partial f_1}{\partial x_1}(x) - \frac{\partial f_2}{\partial x_2}(x) + \dots + (-1)^{n-1} \frac{\partial f_n}{\partial x_n}(x))\, dx_1 \wedge \dots \wedge dx_n$.

Without loss of generality $\operatorname{supp} \omega \subset [-1, 0]^n$. Then for $i = 2, \dots, n$ one has $\int_{\mathbb{R}^n_-} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \dots \wedge dx_n = \int_{[-1,0]^{n-1}} \left(\int_{-1}^0 \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \dots \widehat{dx_i} \dots dx_n = \int_{[-1,0]^{n-1}} (f_i(x_1, \dots, 0, \dots, x_n) - f_i(x_1, \dots, -1, \dots, x_n) dx_1 \dots \widehat{dx_i} \dots dx_n = 0$, and for $i = 1$ one has in a similar manner $\int_{\mathbb{R}^n_-} \frac{\partial f_1}{\partial x_1} dx_2 \wedge \dots \wedge dx_n = \int_{[-1,0]^{n-1}} (f_1(0, x_2, \dots, x_n) - f_n(-1, x_2, \dots, x_n))\, dx_2 \dots dx_n = \int_{\mathbb{R}^{n-1}} f_n(0, x_2, \dots, x_n) dx_2 \dots dx_n = \int_{\partial \mathbb{R}^n_-} \iota^* \omega$.

For an arbitrary M let $\{(U_\alpha, V_\alpha, x_\alpha)\}$ be an oriented locally finite atlas, and $\{\varrho_\alpha\}$ be a partition of unity subordinate to $\{(U_\alpha\}$. The support $\operatorname{supp} \omega$ is compact, so it intersects only a finite number of charts: $\operatorname{supp} \omega \subset U_1 \cup \dots \cup U_N$ and $\operatorname{supp} \omega \cap U_\alpha = \varnothing$ for $\alpha \ne 1, \dots, N$. Then $\omega = \sum_{i=1}^N \varrho_i \omega$. By linearity it suffices to prove the theorem for every form $\omega_i \stackrel{\text{def}}{=} \varrho_i \omega$, $i = 1, \dots, N$.

Now $\int_M d\omega_i = \int_{V_i} (x_i^{-1})^* d\omega_i = \int_{V_i} d(x_i^{-1})^* \omega_i$ (due to naturality of d) $= \int_{V_i \cap \mathbb{R}^{n-1}} (x_i^{-1})^* \omega_i$ (proved above) $= \int_{\partial M} \omega_i$. □

Example 2. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, and $\nu \stackrel{\text{def}}{=} f(z)\,dz \stackrel{\text{def}}{=} f(z)(dx + idy)$ be a 1-form with complex values ($z = x + iy$ where $x, y \in \mathbb{R}$ are coordinates on $\mathbb{C} = \mathbb{R}^2$ viewed as a (real) 2-manifold). If $f(x+iy) \stackrel{\text{def}}{=} g(x,y) + ih(x,y)$ where $g, h : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions then $\nu = (gdx - hdy) + i(gdy + hdx)$ and $d\nu = \left((-\frac{\partial g}{\partial y} - \frac{\partial h}{\partial x}) + i(\frac{\partial g}{\partial x} - \frac{\partial h}{\partial y})\right) dx \wedge dy = 0$ by the Cauchy–Riemann theorem. Let $M \subset \mathbb{C}$ be diffeomorphic to a disc, with its boundary $\partial M = \gamma([0,1])$ where $\gamma : [0,1] \to \mathbb{C}$ is a smooth curve. Then $\int_{\partial M} f(z)\,dz = \int_0^1 \gamma^*(f(z)\,dz) = \int_0^1 f(\gamma(t))\gamma'(t)\,dt = \int_M d\nu = 0$ — one of the principal theorems of elementary complex analysis.

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic in $\mathbb{C} \setminus \{0\}$, and $M \subset \mathbb{C} \setminus \{0\}$ is diffeomorphic to an annulus, with its boundary $\partial M = \gamma_1([0,1]) \cup \gamma_2([0,1])$ where $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ are smooth curves. Like in the previous case one obtains $0 = \int_{\partial M} f(z)\,dz = \int_0^1 \gamma_1^*(f(z)\,dz) - \int_0^1 \gamma_2^*(f(z)\,dz)$ (the minus sign is due to orienation). So, $\int_0^1 \gamma^*(f(z)\,dz)$ is the same for all curves γ encircling the origin in the counterclockwise direction. This number (not necessarily zero!) divided by $2\pi i$ is called the residue of the 1-form $f(z)\,dz$ at the origin.