LECTURE 6

ABSTRACT. Operations on vector bundles. Differential forms. Orientation.

1. Operations on vector bundles.

1.1. *Dual bundle.* Let E be a rank n vector bundle on the base B, $B = \bigcup_\alpha U_\alpha$, a trivialization, and $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$, transition maps. Consider maps $\widetilde{\psi}_{\alpha\beta} \stackrel{\mathrm{def}}{=} \psi_{\beta\alpha}^T = \left((\psi_{\alpha\beta})^{-1}\right)^T$ on the same trivialization (T means a transposed matrix).

Proposition 1. *Maps $\widetilde{\psi}_{\alpha\beta}$ satisfy the cocycle condition.*

Proof. $\widetilde{\psi}_{\alpha\alpha} = \psi_{\alpha\alpha}^T = I^T = I$ (the identity matrix). $\widetilde{\psi}_{\alpha\beta}\widetilde{\psi}_{\beta\gamma} = \psi_{\beta\alpha}^T\psi_{\gamma\beta}^T = (\psi_{\gamma\beta}\psi_{\beta\alpha})^T = \psi_{\gamma\alpha}^T = \widetilde{\psi}_{\alpha\gamma}$. □

So, $\widetilde{\psi}_{\alpha\alpha}$ are transition maps of a rank n vector bundle. This bundle is called a dual bundle to E and is denoted E^*.

For any vector bundle E and a point $a \in B$ of the base we will be denoting $E_a \stackrel{\mathrm{def}}{=} p^{-1}(a)$ the fiber over the point a. Let $v \in E_a$ and $\xi \in E_a^*$. Fix an index α such that $a \in U_\alpha$; then v is represented in the corresponding trivializing chart as (a, x, α) and ξ, as (a, η, α). Define the bilinear form $\langle \cdot, \cdot \rangle : E_a^* \times E_a \to \mathbb{R}$ by $\langle \xi, v \rangle \stackrel{\mathrm{def}}{=} (\eta, x)$ where the braces mean the standard scalar product in $\mathbb{R}^n$ (recall that $x, \eta \in \mathbb{R}^n$ by the definition of the trivialization). If $x \in U_\beta$ then the vectors v and ξ are represented in the corresponding trivialization as (a, x', β), (a, η', β) where $x' = \psi_{\alpha\beta}(a)x$ and $\eta' = \widetilde{\psi}_{\alpha\beta}(a)\eta$. Then the value $\langle \xi, v \rangle$ becomes $(\eta', x') = (\widetilde{\psi}_{\alpha\beta}(a)\eta, \psi_{\alpha\beta}(a)x) = (\psi_{\alpha\beta}(a)^T\psi_{\beta\alpha}(a)^T\eta, x) = (\eta, x)$, that is, does not change. So, the value $\langle \xi, v \rangle$ is well-defined and does not depend on the choice of a trivializing chart. Since the standard scalar product in $\mathbb{R}^n$ is a nondegenerate biliniar form, the form $\langle \cdot, \cdot \rangle : E_a^* \times E_a \to \mathbb{R}$ also is. So, one can understand E_a^*, for every $a \in B$, as a dual space to the vector space E_a, as the notation suggests.

For a manifold M the bundle dual to the tangent bundle TM is called a contangent bundle and is denoted T^*M.

Theorem 1. *Any real vector bundle on a manifold is equivalent to its dual.*

Proof. Since E_a^* is dual to E_a for every a, a linear isomorphism $R_a : E_a \to E_a^*$ is the same as a nondegenerate bilinear form B_a on E_a: $B_a(u, v) \stackrel{\mathrm{def}}{=} \langle R_a(u), v \rangle$, and vice versa, if B_a is known then $R_a(u) : E_a \to \mathbb{R}$ is defined as $R_a(u)v \stackrel{\mathrm{def}}{=} B_a(u, v)$. If a bilinear form B_a is symmetric then it is the same as a quadratic form Q_a on E_a: $Q_a(u) = B_a(u, u)$, and vice versa, if Q_a is known then a symmetric B_a is defined by $B_a(u, v) = \frac{1}{2}(Q_a(u+v) - Q_a(u) - Q_a(v))$. So, to prove the theorem it is enough to define a nondegenerate quadratic form Q_a on every E_a, which depend on a continuously (or smoothly, if smooth bundles are considered).

If $a \in U_\alpha$ (a trivializing chart for E and E^*) and $v \in E_a$ is represented by a triple (a, x, α) in the corresponding trivialization, then take $Q_a^{(\alpha)}(v) \stackrel{\mathrm{def}}{=} (x, x)$ where braces mean the standard scalar product in $\mathbb{R}^n$. Let then ϱ_α be a partition of uniry subordinate to the cover $M = \bigcup_\alpha U_\alpha$. Define $Q_a(v) \stackrel{\mathrm{def}}{=} \sum_\alpha \varrho_\alpha(a)Q_a^{(\alpha)}(v)$. Since the form $Q_a^{(\alpha)}$ is positive definite, $\varrho_\alpha(a) \geq 0$ and $\sum_\alpha \varrho_\alpha(a) = 1$, the form Q_a is positive definite and therefore nondegenerate. □

By Theorem 1 the contangent bundle T^*M of a manifold M is equivalent to its tangent bundle. Nevertheless the properties of the operation (a functor) T^* relating to a manifold its cotangent bundle are quite different from those of the operation (a functor) T relating to a manifold its tangent bundle. In particular, if $f : M_1 \to M_2$ is a smooth map then there is no analog of the bundle morphism $f' : TM_1 \to TM_2$ for cotangent bundles. At the same time, let ν be a section of the cotangent bundle to the manifold M_2 (such section is called a differential 1-form on M_2; the term will be explained below in Section 2). Then define a section $f^*\nu$ of the bundle T^*M_1 as follows: the value of $f^*\nu(a)$ where $a \in M_1$ is the linear functional on T_aM_1 taking on a vector $u \in T_aM_1$ the value $\langle \nu(a), f'(a)u \rangle$. The section $f^*\nu$ is called a pullback of the 1-form ν to the manifold M_1; if M_1 is a submanifold of M_2 and f, a tautological embedding (for $a \in M_1$ one has $f(a) = a \in M_2$), then $f^*\nu$ is called a restriction of the form ν to the submanifold. Note that for vector fields (sections of the tangent bundle) the pullback (and even a restriction) cannot be defined: nor a pushforward. For example, a vector $Z(a)$ in the point $a \in M_1$ of a submanifold $M_1 \subset M_2$ need not be tangent to M_1.

Let M be a manifold, $a \in M$ a point covered by a chart U with the coordinate system $x = (x_1, \dots, x_n)$. A basis in T_a^*M dual to the basis $\frac{\partial}{\partial x_1}(a), \dots, \frac{\partial}{\partial x_n}(a)$ is denoted $dx_1(a), \dots, dx_n(a)$ (dual basis means that $\langle dx_i(a), \frac{\partial}{\partial x_j}(a) \rangle$ is equal to 1 if $i = j$ and to 0 otherwise).

1.2. *Direct sum.* Let E_1, E_2 be vector bundles of ranks n_1, n_2 on the same base B. Without loss of generality one can suppose that they have the same set of trivializing charts U_α (prove!); denote by $\psi_{\alpha\beta}^{(1)}$ and $\psi_{\alpha\beta}^{(2)}$ the corresponding transition maps. For a point $a \in B$ define a map $\psi_{\alpha\beta}(a) \in \mathrm{GL}(n_1+n_2, \mathbb{R})$ as $\psi_{\alpha\beta}(a) \stackrel{\text{def}}{=} \psi_{\alpha\beta}^{(1)}(a) \oplus \psi_{\alpha\beta}^{(2)}(a)$ (a block diagonal matrix $(n_1+n_2) \times (n_1+n_2)$ with the $n_1 \times n_1$-block $\psi_{\alpha\beta}^{(1)}(a)$ in the upper left corner and the $n_2 \times n_2$-block $\psi_{\alpha\beta}^{(2)}(a)$ in the lower right; the remaining matrix elements are 0).

Proposition 2. *The maps $\psi_{\alpha\beta}$ satisfy the cocycle condition and are therefore transition maps of a vector bundle of rank n_1+n_2 denoted by $E_1 \oplus E_2$. The bundle $E = E_1 \oplus E_2$ contains two subbundles isomorphic to E_1 and E_2 and such that $E_a \stackrel{def}{=} (E_1)_a \oplus (E_2)_a$ for every a (fibers of the subbundles on the right-hand side). (In other words, there are bundle morphisms $\iota_1 : E_1 \to E$ and $\iota_2 : E_2 \to E$ which are embeddings on every fiber, and $E_a = \iota_1((E_1)_a) \oplus \iota_2((E_2)_a)$.)*

The proof of Proposition 2 is an exercise and is similar to the proof of Proposition 1 and the reasoning after it.

1.3. *Tensor product.* Let E_1, E_2 be vector bundles of ranks n_1, n_2 on the same base B; suppose again that they have the same trivialization U_α and the transition maps $\psi_{\alpha\beta}^{(1)}$ and $\psi_{\alpha\beta}^{(2)}$. Then define $\psi_{\alpha\beta}(a) = \psi_{\alpha\beta}^{(1)}(a) \otimes \psi_{\alpha\beta}^{(2)}(a) \in \mathrm{GL}(n_1 n_2, \mathbb{R})$ (a matrix $n_1 n_2 \times n_1 n_2$; its elements are indexed by the pairs of pairs of indices $((i,j),(k,\ell))$ as follows: $\psi_{\alpha\beta}(a)_{(i,j),(k,\ell)} = \psi_{\alpha\beta}^{(1)}(a)_{ik}\psi_{\alpha\beta}^{(2)}(a)_{jl}$; here $i,k = 1,\dots,n_1$ and $j,\ell = 1,\dots,n_2$.

Proposition 3. *The maps $\psi_{\alpha\beta}$ satisfy the cocycle condition and are therefore transition maps of a vector bundle of rank $n_1 n_2$ denoted by $E_1 \otimes E_2$. The fiber $(E_1 \otimes E_2)_a$ is isomorphic to $(E_1)_a \otimes (E_2)_a$ for every a; this isomorphism can be made smooth on a.*

Again, the proof is an exercise.

Example 1. Let $M = \mathbb{R}P^1 = S^1$ and E be a tautological (Moebius) bundle of rank 1 on M (see Problem 1 of Set 3). Its trivialization consists of two charts, $U_a = S^1 \setminus \{a\}$ and $U_b = S^1 \setminus \{b\}$ where a and b are opposite (or any two distinct) points in S^1. The intersection $U_a \cap U_b$ consists of two arcs, A_1 and A_2, and the transition map $\psi(q) \in \mathrm{GL}(1,\mathbb{R}) = \mathbb{R} \setminus \{0\}$ is $\psi(q) = 1$ for $q \in A_1$ and $\psi(q) = -1$ for $q \in A_2$. Then the transition map for the tensor square $E^{\otimes 2} = E \otimes E$ is $\psi(q)^2 = 1$ for every q. Thus, the rank 1 bundle $E^{\otimes 2}$ is trivial. The bundle $E^{\otimes n}$ is trivial for n even and isomorphic to E for n odd.

1.4. *External (wedge) power.* Let E be a rank n vector bundle, and $1 \le k \le n$, an integer. Let $\{U_\alpha\}$ be a trivialization, and $\psi_{\alpha\beta} \in \mathrm{GL}(n,\mathbb{R})$, transition maps. Define $\widetilde{\psi}_{\alpha\beta}(a) = (\varphi_{\alpha\beta}(a))^{\wedge k}$ (if A is a $(n \times n)$-matrix then $A^{\wedge k}$ is a $(\binom{n}{k} \times \binom{n}{k})$-matrix with rows and columns indexed by increasing sequences $1 \le i_1 < \dots < i_k \le n$; the matrix element corresponding to $(i_1,\dots,i_k)$ and $(j_1,\dots,j_k)$ is equal to the determinant of the submatrix of A formed by the rows $i_1,\dots,i_k$ and the columns $j_1,\dots,j_k$. If A is a matrix of a linear operator $V \to V$ then $A^{\wedge k}$ is a matrix of the linear operator $\Lambda^k V \to \Lambda^k V$ mapping every decomposable element $v_1 \wedge \dots \wedge v_k$ to $Av_1 \wedge \dots \wedge Av_k$.)

Proposition 4. *The maps $\widetilde{\psi}_{\alpha\beta}$ satisfy the cocycle condition and are therefore transition maps of a vector bundle of rank $\binom{n}{k}$ denoted by $\Lambda^k E$. The fiber $(\Lambda^k E)_a$ is the k-th external power $\Lambda^k E_a$.*

The proof is an exercise (hint: $(AB)^{\wedge k} = A^{\wedge k} B^{\wedge k}$).

2. **k-forms and a differential.** Let M be a n-dimensional manifold, and $1 \le k \le n$, an integer. A section of the bundle $\Lambda^k T^* M$ is called a differential k-form. Let $a \in U$ where (U,V,x) is a coordinate system. A basis of the space $\Lambda^k T_a^* M$ is formed by the wedge products $dx_{i_1} \wedge \dots \wedge dx_{i_k}$ where $1 \le i_1 < \dots < i_k \le n$, and $dx_1, \dots, dx_n$ is the standard basis in $T_a^* M$. Thus in coordinates a k-form looks like $\sum_{1 \le i_1 < \dots < i_k \le n} \omega_{i_1,\dots,i_k}(a)\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$. The set of k-forms on M is sometimes denoted by $\Omega^k(M)$; it is an infinite-dimensional vector space and naturally a module over the algebra $C^\infty(M)$.

It follows from Proposition 4 and Proposition 1 that the fiber $\Lambda^k T_a^* M$ of the bundle is the space of k-linear skew-symmetric forms on the tangent space $T_a M$. If $f : M_1 \to M_2$ is a smooth map, then one can define a pullback $f^*\omega$ of a k-form ω on M_2 as a k-form on M_1 such that the value of $f^*\omega(a)$ on vectors $v_1, \dots, v_k \in T_a M_1$ is equal to $\omega(a)(f'(a)v_1, \dots, f'(a)v_k)$. The pullback is a linear operation.

Example 2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Then for the 1-form $f^* dx_i$ and any $j = 1, \dots, n$ one has

$$\langle f^* dx_i, \frac{\partial}{\partial x_j} \rangle = \langle dx_i, f' \frac{\partial}{\partial x_j} \rangle = \langle dx_i, \sum_{p=1}^n \frac{\partial f_p}{\partial x_j} \frac{\partial}{\partial x_p} \rangle = \frac{\partial f_i}{\partial x_j}.$$

Hence, $f^* dx_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}\, dx_j$.

If V is a n-dimensional vector space then the direct sum $\Lambda V = \bigoplus_{k=0}^{n} V^{\wedge k}$ is an algebra (called the exterior algebra of V) with respect to the exterior multiplication of forms. Namely, elements of $V^{\wedge k}$ are k-linear skew-symmetric forms on V^*, and the wedge product of the k_1-form u and the k_2-form v is defined as a (k_1+k_2)-form $u \wedge v$ such that

$$(u \wedge v)(\xi_1, \dots, \xi_{k_1+k_2}) \quad \sum_{\substack{\{1,\dots,k_1+k_2\}=I \sqcup J \\ I=\{i_1<\dots<i_{k_1}\}, J=\{j_1<\dots<j_{k_2}\}}} \quad (-1)^{i_1+\dots+i_{k_1}-k_1(k_1+1)/2} u(\xi_{i_1}, \dots, \xi_{i_{k_1}}) v(\xi_{j_1}, \dots, \xi_{j_{k_2}}).$$

Proposition 5. *The exterior algebra is associative and super-commutative:* $(u \wedge v) \wedge w = u \wedge (v \wedge w)$ *and* $v \wedge u = (-1)^{k_1 k_2} u \wedge v$ *for all* $u \in \Lambda^{k_1} V$, $v \in \Lambda^{k_2} V$, $w \in \Lambda^{k_3} V$.

The proof is a standard exercise in linear algebra. The 0-th exterior power of V is by definition a one-dimensional space equal to $\mathbb{R}$.

So, the direct sum $\bigoplus_{k=0}^{n} \Omega^k(M)$ is an graded associative super-commutative algebra, with the 0-th component being equal to $C^\infty(M)$. A graded algebra is naturally a module over its 0-th component (because if $A = \bigoplus_{k=0}^{m} A_k$ then the product of an element $x \in A^0$ and an element $y \in A^k$ is $xy \in A^k$), so $\Omega^k(M)$ is a $C^\infty(M)$-module.

Example 3. Let $S^n \subset \mathbb{R}^{n+1}$ be a unit sphere. Define a skew-symmetric n-form $\omega(a)$ on $T_a S^n$ as follows: $\omega(a)(v_1, \dots, v_n) = \det(a, v_1, \dots, v_n)$ where $a = (a_1, \dots, a_{n+1}) \in S^n$ and $v_i = (v_{i1}, \dots, v_{i,n+1}) \in \mathbb{R}^{n+1}$, $i = 1, \dots, n$, are vectors tangent to S^n (that is, normal to a) at a. Thus one defines a n-form on S^n called the volume form. The value $\omega(a)$ of the volume form is nondegenerate for any a.

Proposition 6. *The exterior product is a natural operation, that is, it commutes with pullbacks:* $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$.

Proof is an exercise.

Definition 1. Let $f \in C^\infty(M)$ be a smooth function on M. By df one denotes a 1-form such that the value $\langle df(a), v \rangle$ of the linear functional $df(a) : T_a M \to \mathbb{R}$ on a vector $v \in T_a M$ is equal to $v(f)$.

(Recall that a vector $v \in T_a M$ is a linear functional on $C^\infty(M)$.)

Example 4. Let $M = \mathbb{R}^n$ and $a \in M$; then f becomes a smooth function of n real variables. For a vector $v = \frac{\partial}{\partial x_i}$ one has $v(f) = \frac{\partial f}{\partial x_i}(a) = \langle df(a), v \rangle$, which gives for $df(a)$ the expression $df(a) = \frac{\partial f}{\partial x_1}(a)\, dx_1 + \dots + \frac{\partial f}{\partial x_n}(a)\, dx_n$.

Theorem 2. *There exists a unique linear operator* $d : \Omega^k(M) \to \Omega^{k+1}(M)$ *with the following properties:*

(1) *For* $k = 0$ *it is given by Definition 1.*
(2) *It is a super-derivation: if* $u \in \Omega^{k_1}(M)$ *and* $v \in \Omega^{k_2}(M)$ *then* $d(u \wedge v) = du \wedge v + (-1)^{k_1} u \wedge dv$.

This operator commutes with pullbacks $df^*\omega = f^* d\omega$ *and satisfies the equality* $d^2 = 0$.

The operator d is called exterior differentiation; the form $d\omega$ is called the exterior derivative of the form ω.

Proof. Prove the theorem first for the manifold $M = \mathbb{R}^n$. The covectors $dx_{i_1} \wedge \dots \wedge dx_{i_k}$ for all $1 \le i_1 < \dots < i_k \le n$ form a basis in $\Lambda^k T_a \mathbb{R}^n = \Lambda^k \mathbb{R}^n$ for all $a \in \mathbb{R}^n$, and therefore any k-form is given by the formula $\omega = \sum_{1 \le i_1 < \dots < i_k \le n} \nu_{i_1,\dots,i_k}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$. Take by definition

$$d\omega \stackrel{\text{def}}{=} \sum_{1 \le i_1 < \dots < i_k \le n} d\nu_{i_1,\dots,i_k}(x) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum_{i=1}^{n} \sum_{1 \le i_1 < \dots < i_k \le n} \frac{\partial \nu(x)}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

For 0-forms the operator d coincides with that of Definition 1, cf. Example 4. Equality $d(u \wedge v) = du \wedge v + (-1)^{k_1} u \wedge dv$ for any u and v is checked by a straightforward computation.

Prove that $d^2 = 0$. Let first $k = 0$; then $dd\nu(x) = d(\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} dx_i) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} dx_j \wedge dx_i$. Since $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$ and $dx_j \wedge dx_i = -dx_i \wedge dx_j$, the terms in the last sum cancel pairwise, and $d^2 \nu(x) = 0$. Then take $k = 1$, so $\omega = \sum_{i=1}^{n} \nu_i(x)\, dx_i$. Then $d\omega = \sum_{i,j=1}^{n} \frac{\partial \nu_i}{\partial x_j} dx_j \wedge dx_i$ and $dd\omega = \sum_{i,j,k=1}^{n} \frac{\partial^2 \nu_i}{\partial x_j \partial x_k} dx_k \wedge dx_j \wedge dx_i = 0$ by the same reason. Now by the super-derivation property for any k one has $dd\omega = d(\sum_{i_1 < \dots < i_k} d\nu_{i_1,\dots,i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}) = \sum_{i_1 < \dots < i_k} d^2 \nu_{i_1,\dots,i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} - d\nu_{i_1,\dots,i_k} \wedge d^2 x_{i_1} \wedge \dots \wedge dx_{i_k} + \dots = 0$.

Prove that the operation d is *natural*, that is, commutes with pullbacks. Consider a smooth map $f : M_1 = \mathbb{R}^n \to M_2 = \mathbb{R}^m$ given by the formula $f(x) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$. Again, let first $k = 0$, so that $\omega = \nu(y)$ (a function of the argument $y \in \mathbb{R}^m$), and $d\omega = \sum_{i=1}^{m} \frac{\partial \nu}{\partial y_i} dy_i$. One has $f^*\omega = \nu(f(x))$, and $df^*\omega = \sum_{i=1}^{n} \frac{\partial \nu(f(x))}{\partial x_i} dx_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial \nu}{\partial y_j}(f(x)) \frac{\partial f_j}{\partial x_i}(x)\, dx_i$. By Example 2 and Proposition 6 one has $f^* d\omega = \sum_{i=1}^{n} f^* \frac{\partial \nu}{\partial x_i} \wedge f^* dx_i = \sum_{i=1}^{n} \frac{\partial \nu}{\partial x_i}(f(y)) \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x)\, dx_j = df^*\omega$.

For $k = 1$ the reasoning is similar: $\omega = \sum_{i=1}^m \nu_i(y)\, dy_i$ and $d\omega = \sum_{i,j=1}^m \frac{\partial \nu_i}{\partial y_j}\, dy_j \wedge dy_i$. Then by Proposition 6 $f^*\omega = \sum_{i=1}^m \sum_{p=1}^n \nu_i(f(y)) \frac{\partial f_i}{\partial x_p}\, dx_p$; so

$$\begin{aligned}
df^*\omega &= \sum_{i,j=1}^m \sum_{p,q=1}^n \frac{\partial \nu_i}{\partial y_j}(f(x)) \frac{\partial f_j}{\partial x_q}(x) \frac{\partial f_i}{\partial x_p}(x)\, dx_q \wedge dx_p + \sum_{i=1}^m \sum_{p,q=1}^n \nu_i(f(y)) \frac{\partial^2 f_i}{\partial x_p \partial x_q}(x)\, dx_q \wedge dx_p \\
&= \sum_{i,j=1}^m \sum_{p,q=1}^n \frac{\partial \nu_i}{\partial y_j}(f(x)) \frac{\partial f_j}{\partial x_q}(x) \frac{\partial f_i}{\partial x_p}(x)\, dx_q \wedge dx_p + \sum_{i=1}^m \sum_{1 \le p < q \le n} \frac{\partial^2 f_i}{\partial x_p \partial x_q}(x)(dx_q \wedge dx_p + dx_p \wedge dx_q) \\
&= \sum_{i,j=1}^m \sum_{p,q=1}^n \frac{\partial \nu_i}{\partial y_j}(f(x)) \frac{\partial f_j}{\partial x_q}(x) \frac{\partial f_i}{\partial x_p}(x)\, dx_q \wedge dx_p.
\end{aligned}$$

At the same time by Proposition 6 and Example 4

$$f^* d\omega = \sum_{i,j=1}^m f^* \frac{\partial \nu_i}{\partial y_j} \wedge f^* dy_j \wedge f^* dy_i = \sum_{i,j=1}^m \sum_{p,q=1}^n \frac{\partial \nu_i}{\partial y_j}(f(x)) \frac{\partial f_j}{\partial x_q}(x) \frac{\partial f_i}{\partial x_p}(x)\, dx_q \wedge dx_p = df^*\omega.$$

For an arbitrary k it suffices (because f^* is a linear map) to check naturality of d for the form $\omega = \nu(x) dx_{i_1} \wedge \dots \wedge dx_{i_k}$. Then $d\omega = d\nu \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$, so by Proposition 6 one has $f^* d\omega = f^* d\nu \wedge f^* dx_{i_1} \wedge \dots \wedge f^* dx_{i_k}$ and also $f^*\omega = (f^*\nu) f^* dx_{i_1} \wedge \dots \wedge f^* dx_{i_k}$. The operator d has been proved to be natural for 0-forms (functions) and 1-forms, to satisfy $d^2 = 0$ and to be a super-derivation, so one has

$$\begin{aligned}
df^*\omega &= d(f^*\nu) \wedge f^* dx_{i_1} \wedge \dots \wedge f^* dx_{i_k} + \sum_{s=1}^k (-1)^{s+1} (f^*\nu) \wedge f^* dx_{i_1} \wedge \dots \wedge df^* dx_{i_s} \wedge \dots \wedge f^* dx_{i_k} \\
&= f^* d\nu \wedge f^* dx_{i_1} \wedge \dots \wedge f^* dx_{i_k} + \sum_{s=1}^k (-1)^{s+1} (f^*\nu) \wedge f^* dx_{i_1} \wedge \dots \wedge f^* d^2 x_{i_s} \wedge \dots \wedge f^* dx_{i_k} \\
&= f^* d\nu \wedge f^* dx_{i_1} \wedge \dots \wedge f^* dx_{i_k} = f^* d\omega
\end{aligned}$$

Now prove the theorem for an arbitrary manifold M. Let $a \in U \subset M$ where (U, V, x) is the system of coordinates. For a k-form ω on M let $\omega_x \stackrel{\text{def}}{=} (x^{-1})^* \omega|_U$; it is a k-form on $V \subset \mathbb{R}^n$. Take by definition $d\omega \stackrel{\text{def}}{=} x^*(d\omega_x)$; here ω_x is a k-form on an open subset of $\mathbb{R}^n$, so $d\omega_x$ was defined earlier.

Check now that $d\omega$ does not depend on the choice of coordinates. Indeed, let $(\tilde{U}, \tilde{V}, y)$ be another system of coordinates and $a \in U \cap \tilde{U}$, with $\varphi = y \circ x^{-1} : W \to \tilde{W}$ being a transition map ($W \subset V$ and $\tilde{W} \subset \tilde{V}$ are open subsets). Then $\omega_x = \varphi^* \omega_y$, so $d\omega_x = \varphi^* d\omega_y$: φ is a map between open subsets of $\mathbb{R}^n$, so the naturality of d with respect to such maps is already proved. Therefore the $d\omega$ defined by means of y is $(y^{-1})^* \varphi^* d\omega_x = (y^{-1})^* y^* (x^{-1})^* d\omega_x = (x^{-1})^* d\omega_x = d\omega$.

So, the exterior differentiation operator d is well-defined on any manifold. Properties of the operator d on arbitrary manifolds follow immediately from the definition and the corresponding properties of d on $\mathbb{R}^n$; details of proofs are left to the reader. $\square$

3. **Orientation of a manifold.** An atlas $\{(U_\alpha, V_\alpha, x_\alpha)\}$ on a manifold M is called oriented if for any α, β and for any $u \in x_\alpha(U_\alpha \cap U_\beta)$ one has $\det \varphi'_{\alpha\beta}(u) > 0$. (Recall that the transition map $\varphi_{\alpha\beta}$ is smooth and has an inverse map $\varphi_{\beta\alpha}$, so that $\varphi'_{\alpha\beta}(u) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, too. Hence $\varphi'_{\alpha\beta}(u) \neq 0$ for any atlas; orientation chooses the sign here.) Two oriented atlases are called equivalent if their union is an oriented atlas, too. The equivalence class of oriented atlases on M is called an orientation of M. A manifold having an oriented atlas is called orientable; a manifold with the orientation chosen is called oriented.

Remark. A connected orientable manifold has exactly two orientations. The proof is a not-so-easy exercise.

Example 5. A circle S^1 has an atlas of two coordinate systems, $(U_a, (-\pi, \pi), x)$ and $(U_b, (0, 2\pi), x)$ where a, b are two opposite points, $U_a \stackrel{\text{def}}{=} S^1 \setminus \{a\}$ and $U_b \stackrel{\text{def}}{=} S^1 \setminus \{b\}$, and the coordinates x and y are polar angles counted from the point b. The intersection $U_a \cap U_b$ consists of two arcs, ab and ba, and the (only one) transition map is $\varphi_{ab}(u) = u$ on the arc ba and $\varphi_{ab} = u + 2\pi$ on the arc ab. Thus one has $\det \varphi'_{ab}(u) = 1$ everywhere and the atlas is oriented. If $(U_c, (-\pi, \pi), z)$ is the third chart with $z = -x$ and $U_c = U_a$, then $\varphi_{ac}(u) = -u$, and $\det \varphi'_{ac}(u) = -1$. So the three-chart atlas is not oriented, unlike the two-chart one.

Example 6. The Moebius band M is the square $[0, 1] \times (0, 1)$ with the identifications $(0, t) \sim (1, 1 - t)$ for all $t \in (0, 1)$. It has an atlas of two charts: $U_1 = V_1 = (0, 1) \times (0, 1) \subset M$ and the coordinate map is the identity map $(x_1(a, b) = a, x_2(a, b) = b)$, and the other is $U_2 = \{(a, b) \in M \mid a \neq 1/2\}$, and the map is $y_1(a, b) = a, y_2(a, b) = b$ for $a > 1/2$ and $y_1(a, b) = a + 1, y_2(a, b) = 1 - b$ for $a < 1/2$. Thus one has $\det \varphi'_{12}(a, b) = 1$ for $a > 1/2$ and $\det \varphi'_{12}(a, b) = -1$ for $a < 1/2$. So the atlas is not oriented. In fact, the Moebius band with the standard smooth structure does not have an oriented atlas (is not orientable); we do not give the proof of this fact here.