LECTURE 5

Abstract. Phase curves and commutators of vector fields.

1. **Phase curves of a vector field.** A curve $\gamma : \mathbb{R} \to M$ on a smooth manifold M is called an phase curve of a vector field Z if $\gamma'(t) = Z(\gamma(t))$ for all $t \in \mathbb{R}$. One can also consider phase curves defined on open subsets of $\mathbb{R}$, say, segments (a, b).

Theorem 1. *Let M be a manifold and Z, a smooth vector field on it. Then for any $a \in M$ there exists an open subset $U \subset M$, $a \in U$, and a number $\varepsilon > 0$ such that for any $b \in U$ there exists a unique phase curve $\gamma_b : (-\varepsilon, \varepsilon) \to M$ of the vector field Z such that $\gamma_b(0) = b$. The map $\Gamma : U \times (-\varepsilon, \varepsilon) \to M$ given by a formula $\Gamma(b, t) \stackrel{def}{=} \gamma_b(t)$ is smooth.*

Uniqueness in this theorem means that if $\gamma_1 : (-\varepsilon_1, \varepsilon_1) \to M$ and $\gamma_2 : (-\varepsilon_2, \varepsilon_2) \to M$ are phase curves with $\gamma_1(0) = \gamma_2(0)$ then they coincide where they both are defined, that is, $\gamma_1(t) = \gamma_2(t)$ for any $t \in (-\varepsilon_1, \varepsilon_1) \cap (-\varepsilon_2, \varepsilon_2)$.

Proof. Since it is a local statement, one can suppose that $M = \mathbb{R}^n$. Then the vector field Z is $Z(x) = \sum_{i=1}^n z_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}$, and the curve $\gamma(t) \stackrel{\text{def}}{=} (\gamma_1(t), \dots, \gamma_n(t))$ is its phase curve if the functions $\gamma_1(t), \dots, \gamma_n(t)$ are solutions of the system of ordinary differential equations (ODE) $\frac{d}{dt}\gamma_i(t) = z_i(\gamma_1(t), \dots, \gamma_n(t))$, $i = 1, \dots, n$. The theorem follows now from the existnce and uniqueness theorem for systems of the first-order ODE. □

If M is compact, the situation becomes global:

Theorem 2. *Let M be a compact manifold and Z, a smooth vector field on it. Then there exists a smooth map $\Gamma : M \times \mathbb{R} \to M$ (called a phase flow of the vector field Z) such that*

(1) *$\Phi(a, 0) = a$ for all $a \in M$.*
(2) *$\Phi(\Phi(a, t_1), t_2) = \Phi(a, t_1 + t_2)$ for all $a \in M$ and $t_1, t_2 \in \mathbb{R}$.*
(3) *$\left.\frac{d\Phi(a,t)}{dt}\right|_{t=\tau} = Z(\Phi(a, \tau))$.*

Taking $t_1 = t$ and $t_2 = -t$ in Property 2, one obtains $\Phi(\Phi(a, t), -t) = \Phi(a, 0) = a$ (from Property 1). Thus, for any fixed $t \in \mathbb{R}$ the map $\Phi_t : M \to M$ defined by $\Phi_t(a) = \Phi(a, t)$ is a diffeomorphism (it is smooth because Φ is smooth and its inverse is Φ_{-t}, which is also smooth). Thus, Properties 1 and 2 (known together as 1-parametric group property) can be expressed by saying that $t \mapsto \Phi_t$ is a homomorphism from the additive group $\mathbb{R}$ (the operation is addition of numbers) to the group of diffeomorphisms of the manifold M (the operation is composition of maps).

Proof of Theorem 2. Take for every $a \in M$ an open set $U_a \subset M$ and a number $\varepsilon_a > 0$ as in Theorem 1. By the compactness of M there exist points $a_1, \dots, a_N \in M$ such that $M = U_{a_1} \cup \dots \cup M_{a_N}$; take $\varepsilon \stackrel{\text{def}}{=} \min(\varepsilon_{a_1}, \dots, \varepsilon_{a_N}) > 0$. Then for any point $a \in M$ there exists (and is unique) a phase curve $\gamma_a : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = a$. For any $s \in (-\varepsilon, \varepsilon)$ the curve $\gamma_{a,s}(t) \stackrel{\text{def}}{=} \gamma_a(t + s)$ is a phase curve of Z defined on an interval $(-\varepsilon + s, \varepsilon - s)$ and such that $\gamma_{a,s}(0) = \gamma_a(s)$. Due to uniqueness, $\gamma_{a.s}(t) = \gamma_{\gamma_a(s)}(t)$ for all possible t. This allows to extend γ_a to the smooth curve $(-\varepsilon, 3\varepsilon/2) \to M$ (to be called still γ_a saying that $\gamma_a(t)$ is defined as it previously was for $|t| < \varepsilon$ and $\gamma_a(t) \stackrel{\text{def}}{=} \gamma_{\gamma_a(\varepsilon/2)}(t - \varepsilon/2)$ if $\varepsilon/2 < |t| < 3\varepsilon/2$; the curve obtained is again an phase curve of Z. In a similar manner one may extend γ_a to an phase curve $(-3\varepsilon/2, 3\varepsilon/2) \to M$, etc., proving eventually that for any $a \in M$ there exists a unique phase curve $\gamma_a : \mathbb{R} \to M$ of the vector field Z.

Define now $\Phi(a, t) \stackrel{\text{def}}{=} \gamma_a(t)$. Property 1 is obvious; Property 3 expresses the fact that γ_a is a phase curve. Property 2 follows from uniquenes of the phase curve by the trick already used in the previous paragraph: for any fixed t_1 the curve $\delta(t) \stackrel{\text{def}}{=} \Phi(a, t_1 + t)$ is a phase curve of Z with $\delta(0) = \Phi(a, t_1)$. By uniqueness, $\delta(t) = \Phi(\Phi(a, t_1), t)$ for any t; take $t = t_2$. □

Example 1. A phase curve of the vector field $\frac{\partial}{\partial x_1}$ on $\mathbb{R}^n$ passing through a point $a = (a_1, \dots, a_n)$ is $\gamma_a(t) = (a_1 + t, a_2, \dots, a_n)$; it is defined for all $t \in \mathbb{R}$. An phase curve of the vector field $x_1 \frac{\partial}{\partial x_1} + \dots + x_n \frac{\partial}{\partial x_n}$ passing through a point a is $\gamma_a(t) = ae^t$, $t \in \mathbb{R}$.

Example 2. Phase curves of the vector field $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$ are $\gamma(t) = r(\cos(t + \varphi), \sin(t + \varphi))$ for all $r \geq 0$ and $\varphi \in \mathbb{R}$. They are defined for all $t \in \mathbb{R}$.

Example 3. A phase curve of the vector field $t^2\frac{d}{dt}$ passing through a point $a \in \mathbb{R}$ is $\gamma_a(t) = \frac{a}{1-at}$. These curves (except $\gamma_0 \equiv 0$) are defined on infinite segments: $t \in (-\infty, 1/a)$ for $a > 0$ and $t = (1/a, +\infty)$ for $a < 0$. So, there is no $\varepsilon > 0$ such that all γ_a are defined on $(-\varepsilon, \varepsilon)$. For any segment $(p, q) \subset \mathbb{R}$, though, there exists such ε that γ_a for all $a \in (p, q)$ are defined on $(-\varepsilon, \varepsilon)$ (this is a particular case of Theorem 1).

2. Commutator of vector fields.

Theorem 3. *Let $A, B : X \to X$ be derivations of an associative commutative algebra X. Then the commutator $C \stackrel{def}{=} [A, B] \stackrel{def}{=} A \circ B - B \circ A$ is also a derivation ($\circ$ meaning composition of maps).*

Proof. The map C is apparently linear; take now $f, g \in X$, so $C(fg) = A(B(fg)) - B(A(fg)) = A(fB(g)+B(f)g) - B(fA(g)+A(f)g) = A(f)B(g)+fA(B(g))+A(B(f))g+B(f)A(g)-B(f)A(g)-f(B(A(g))-B(A(f))g-A(f)B(g) = fC(g) + C(f)g$. □

Corollary 1. *A commutator of two vector fields on a manifold M is a vector field.*

Here vector fields are considered as derivations of the algebra $C^\infty(M)$ of smooth functions on M. In coordinates $x = (x_1, \dots, x_n)$ if $A = \sum_{i=1}^n a_i(x)\frac{\partial}{\partial x_i}$ and $B = \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i}$ then for any function $f \in C^\infty(M)$ one has

$$[A, B](f) = A(\sum_{i=1}^n b_i(x)\frac{\partial f}{\partial x_i}) - B(\sum_{i=1}^n a_i(x)\frac{\partial f}{\partial x_i}) = \sum_{i,j=1}^n a_j(x)\frac{\partial b_i(x)}{\partial x_j}\frac{\partial f}{\partial x_i} - \sum_{i,j=1}^n b_j(x)\frac{\partial a_i(x)}{\partial x_j}\frac{\partial f}{\partial x_i}$$
$$+ \text{ terms containing } \frac{\partial^2 f}{\partial x_i \partial x_j} \text{ that have to cancel by Corollary 1.}$$

So, in coordinates $[A, B] = \sum_{i=1}^n \left(\sum_{j=1}^n a_j(x)\frac{\partial b_i(x)}{\partial x_j} - b_j(x)\frac{\partial a_i(x)}{\partial x_j}\right)\frac{\partial}{\partial x_i}$.

We are going now to give a geometric description of the commutator of vector fields. Take a point $a \in M$ and let $\gamma_a : (-\varepsilon, \varepsilon) \to M$ be an phase curve of the field A with $\gamma_a(0) = a$; take $P_t \stackrel{\text{def}}{=} \gamma_a(t)$ for any $t \in (-\varepsilon, \varepsilon)$. Let $\delta_{P_t} : (-\varepsilon, \varepsilon) \to M$ be an phase curve of the vector field B with $\delta_{P_t}(0) = P_t$; take $Q_{t,s} \stackrel{\text{def}}{=} \delta_{P_t}(s)$ for any $s \in (-\varepsilon, \varepsilon)$. Then let $\gamma_{Q_{t,s}} : (-\varepsilon, \varepsilon) \to M$ be an phase curve of A with $\gamma_{Q_{t,s}}(0) = Q_{t,s}$; take $R_{t,s} \stackrel{\text{def}}{=} \gamma_{Q_{t,s}}(-t)$ (the same t as before!). Finally, let $\delta_{R_{t,s}} : (-\varepsilon, \varepsilon) \to M$ be an phase curve of the vector field B with $\delta_{R_{t,s}}(0) = R_{t,s}$; take $U(t, s) \stackrel{\text{def}}{=} \delta_{R_{t,s}}(-s)$.

Now $U : (-\varepsilon, \varepsilon)^2 \to M$ is a smooth map; by uniqueness of an phase curve one has $U(t, 0) = U(0, s) = a$ for all $t, s \in (-\varepsilon, \varepsilon)$. Fix $s \in (-\varepsilon, \varepsilon)$; then the formula $\lambda_s(t) \stackrel{\text{def}}{=} U(t, s)$ defines a smooth curve $\lambda_s : (-\varepsilon, \varepsilon) \to M$ with $\lambda_s(0) = a$. The equivalence class of the curve λ_s at a is a vector $\ell(s) \in T_aM$; so $\ell : (-\varepsilon, \varepsilon) \to T_aM$ is a smooth map.

Theorem 4. $\ell'(0) = [A, B](a)$.

Proof. To compute $\ell'(0)$ it is enough to operate with $\ell(s)$ for $s \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ arbitrarily small; to define such $\ell(s)$ one has to know $U(t, s)$ for $t, s \in (-\varepsilon, \varepsilon)$. So, the statement has local nature, and it is enough to prove it for $M = \mathbb{R}^n$ and $a = 0$.

Consider the Taylor's formula for the map U: $U(t, s) = u_{00}+u_{10}t+u_{01}s+u_{11}ts+o(t)+o(s)$, $t, s \to 0$ (pay attention that the last two terms depend on both t and s!). Since $U(t, 0) = U(0, s) = 0$, one has $u_{00} = u_{10} = u_{01} = 0$, and the last two terms are divisible by ts. Thus the curve $\lambda_s(t) = u_{11}ts + so(t) + to(s)$, hence $\ell(s) = \lambda_s'(0) = u_{11}s + o(s)$, and $\ell'(0) = u_{11}$. So, to prove the theorem it suffices to compute u_{11}.

Remember that $A(x) = (a_1(x), \dots, a_n(x))$ and $B(x) = (b_1(x), \dots, b_n(x))$. So, $P_t = (a_1(0)t, \dots, a_n(0)t) + o(t)$. The term $o(t)$ cannot contribute to the term $u_{11}ts$ in the formula for $U(t, s)$ (explain why!), so we will write simply

$P_t = (a_1(0)t, \dots, a_n(0)t) + \dots$, and will be using $\dots$ to denote all such "uninteresting" terms. Now

$$\begin{aligned}
Q_{t,s} &= P_t + (b_1(P_t), \dots, b_n(P_t))s + \dots \\
&= (a_1(0)t + b_1(0)s + \frac{d}{d\tau} b_1(P_\tau)\Big|_{\tau=0} ts + \dots, \dots, a_n(0)t + b_n(0)s + \frac{d}{d\tau} b_n(P_\tau)\Big|_{\tau=0} ts + \dots) \\
&= (a_1(0)t + b_1(0)s + ts \sum_{j=1} a_j(0) \frac{\partial b_1}{\partial x_j}(0) + \dots, \dots, a_n(0)t + b_n(0)s + ts \sum_{j=1} a_j(0) \frac{\partial b_n}{\partial x_j}(0) + \dots), \\
R_{t,s} &= Q_{t,s} - (a_1(Q_{t,s}), \dots, a_n(Q_{t,s})t + \dots \\
&= Q_{t,s} - (a_1(0)t, \dots, a_n(0)t) - (\sum_{j=1}^n b_j(0) \frac{\partial a_1}{\partial x_j}(0)ts + \dots, \dots, \sum_{j=1}^n b_j(0) \frac{\partial a_n}{\partial x_j}(0) + \dots) \\
&= (b_1(0)s, \dots, b_n(0)s) + ts(\sum_{j=1}^n \left(a_j(0) \frac{\partial b_1}{\partial x_j}(0) - b_j(0) \frac{\partial a_1}{\partial x_j}(0) \right), \dots, \sum_{j=1}^n \left(a_j(0) \frac{\partial b_n}{\partial x_j}(0) - b_j(0) \frac{\partial a_n}{\partial x_j}(0) \right)) + \dots, \\
U(t,s) &= R_{t,s} - s(b_1(R_{t,s}), \dots, b_n(R_{t,s})) + \dots = R_{t,s} - s(b_1(0), \dots, b_n(0)) + \dots \\
&= ts(\sum_{j=1}^n \left(a_j(0) \frac{\partial b_1}{\partial x_j}(0) - b_j(0) \frac{\partial a_1}{\partial x_j}(0) \right), \dots, \sum_{j=1}^n \left(a_j(0) \frac{\partial b_n}{\partial x_j}(0) - b_j(0) \frac{\partial a_n}{\partial x_j}(0) \right)) + \dots,
\end{aligned}$$

so the theorem is proved. □

Example 4. A commutator of the vector fields $Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$ is 0. The corresponding phase flows are $\Psi((x,y),t) = (xe^t, ye^t)$ and $\Psi((x,y),s) = (x\cos s + y\sin s, -x\sin s + y\cos s)$ according to Examples 1 and 4. The flows Φ and Ψ commute: $\Phi(\Psi((x,y),s),t) = e^t(x\cos s + y\sin s, -x\sin s + y\cos s) = \Psi(\Phi((x,y),t),s)$. Using the notation from the theorem, the map $U(t,s) \equiv (x,y)$ (a constant map), so $\ell(s) \equiv 0$, confirming the theorem.