LECTURE 4

Abstract. Points and tangent vectors of a manifold via algebra of the smooth functions.

From now on we will consider only manifolds M having a countable (or finite) pre-atlas $\{(U_k, V_k, x_k)\}$, $k = 1, 2, \dots$.

The associative commutative algebra $C^\infty(M)$ consists of smooth maps (functions) $f : M \to \mathbb{R}$; addition and multiplication is performed pointwise. Supply $C^\infty(M)$ with the *compact-open C^∞-topology* such that $f_n \to f$ in this topology would mean uniform convergence $f_n \rightrightarrows f$ on any compact $K \subset M$ together with all partial derivatives (of any order) of all coordinate representations of f. An accurate definition of the topology is left as an exercise.

Let $\{U_\alpha \mid \alpha \in \mathfrak{A}\}$ be a covering of the manifold M by open subsets: $\bigcup_{\alpha\in\mathfrak{A}} U_\alpha = M$. For a function $\varphi : M \to \mathbb{R}$ the set $\operatorname{supp}\varphi \stackrel{\text{def}}{=} \overline{\{b \in M \mid \varphi(b) \neq 0\}}$ is called a support of φ. We say that functions $\varrho_\alpha \in C^\infty(M)$, $\alpha \in \mathfrak{A}$, form a partition of unity subordinate to the cover $\{U_\alpha\}$ if they have the following properties:

(1) $\varrho_\alpha(a) \geq 0$ for all $a \in M$.
(2) $\operatorname{supp}(\varrho_\alpha) \subset U_\alpha$ for every $\alpha \in \mathfrak{A}$.
(3) For every $a \in M$ there exists an open set $U \ni a$ such that the set of all $\alpha \in \mathfrak{A}$ such that $U \cap \operatorname{supp}\varrho_\alpha \neq \varnothing$ is finite.
(4) $\sum_\alpha \varrho_\alpha \equiv 1$.

Note that by Property 3 the sum in the left-hand side of Property 4 is finite for any point $a \in M$ (though for different points it may contain different number of terms, which even need not be bounded). Also, there holds

Corollary 1 (of Property 3). *Let $\{\varrho_\alpha \mid \alpha \in \mathfrak{A}\}$ be a partition of unity, and $K \subset M$ be a compact. Then the set of $\alpha \in \mathfrak{A}$ such that $\operatorname{supp}\varrho_\alpha \cap K \neq \varnothing$ is finite.*

Proof. Suppose the opposite: let $\alpha_1, \alpha_2, \dots \in \mathfrak{A}$ be such that $\operatorname{supp}\varrho_{\alpha_i} \cap K \neq \varnothing$; take a point a_i in the latter intersection. The set $\{a_1, a_2, \dots\} \subset K$ has an accumulation point $a \in K$. By Property 3 there exists a neighbourhood $U \ni a$ that intersects only finitely many sets $\operatorname{supp}\varrho_\alpha$. On the other side, since a is an accumulation point, U contains infinitely many $a_i \in \operatorname{supp}\varrho_{\alpha_i}$. A contradiction. □

The following theorem asserts that the set $C^\infty(M)$ is sufficiently large for any manifold M:

Theorem 1 (partition of unity). *Let manifold M have a countable pre-atlas, and $\{U_\alpha\}$ be an open cover of M. Then there exists a partition of unity subordinate to the cover.*

For proof of this theorem see Exercise Set 4.

Example 1. Let $a, b \in M$, $a \neq b$. Consider a cover $M = U_a \cup U_b$ where $U_a = M \setminus \{a\}$, $U_b = M \setminus \{b\}$, and let ϱ_a, ϱ_b be a subordinate partition of unity. Then $\varrho_a(a) = 0$, $\varrho_b(b) = 0$, and therefore $\varrho_a(b) = 1 - \varrho_b(b) = 1$. So, different points of a manifold are "independent" as zeros of smooth functions: $\varrho(a) = 0$ never implies $\varrho(b) = 0$.

Theorem 2. *Let M be a smooth manifold and $a \in M$. The set $\mathcal{J}_a \stackrel{\text{def}}{=} \{f \in C^\infty(M) \mid f(a) = 0\}$ is a closed maximal ideal of the topological algebra $C^\infty(M)$. Every closed maximal ideal in $C^\infty(M)$ is $\mathcal{J}_a$ for some $a \in M$.*

To prove the theorem we will need two lemmas

Lemma 1. *There exists a sequence of compacts $K_1 \subset K_2 \subset \dots \subset M$ such that $\bigcup_s K_s = M$.*

Proof of this lemma is an exercise, see Exercise Set 4. (Actually, it is necessary to prove Theorem 1, too.)

Lemma 2. *Let $\Phi \subset C^\infty(M)$ be a subset such that its elements have no common zeros: $\forall a \in M \exists \varphi \in \Phi : \varphi(a) \neq 0$. If $\Phi \subset I$ where $I \subset C^\infty(M)$ is a closed ideal then $I = C^\infty(M)$. If the set Φ is finite then the requirement for I to be closed may be omitted.*

Proof. Consider the sets $\Omega_f = \{b \in M \mid f(b) \neq 0\}$ for all $f \in \Phi$; since $f \in \Phi$ have no common zeros, the sets Ω_f form an open cover of M. Let ϱ_f be a partition of unity subordinate to the cover. The function $R = \sum_{f\in\mathcal{J}} \varrho_f f^2$ is defined (the sum is finite in any point by Property 3).

By Lemma 1 and Corollary 1 for any s there is a finite subset $\Phi_s \subset \Phi$ such that $\varrho_\alpha|_{K_s} = 0$ for $\alpha \notin \Phi_s$. Therefore R is a limit point (in the compact-open topology) of the set $\{R_\Psi \stackrel{\text{def}}{=} \sum_{f\in\Psi} \varrho_f f^2 \mid \Psi \subset \Phi \text{ is finite}\}$. Apparently $R_\Psi \in I$ because I is an ideal containing $\Psi \subset \Phi$, and therefore $R \in I$ because I is closed; if Φ is itself finite then $R \in I$ without closedness. The function R is nowhere equal to zero (because $\{\Omega_f\}$ is a cover of M). Therefore $1 = R/R \in \mathcal{J}$, that is, $\mathcal{J}$ is a trivial ideal. □

Proof of Theorem 2. $\mathcal{J}_a$ is obviously an ideal; it is maximal by Example 1 and Lemma 2. If $f_n \to f$ and $f_n \in \mathcal{J}_a$ then $f_n(a) \to f(a)$ as $n \to \infty$ (the one-point set $\{a\}$ is compact), so $f(a) = 0$ and $\mathcal{J}_a$ is closed.

Conversely, let $\mathcal{J} \subset C^\infty(M)$ be a maximal ideal and a closed set, By Lemma 2 there is a point $a \in M$ which is a common zero of all $f \in \mathcal{J}$. This means $\mathcal{J} \subset \mathcal{J}_a$, and therefore $\mathcal{J} = \mathcal{J}_a$ because $\mathcal{J}$ is maximal. $\square$

For any $f \in \mathcal{J}_a$ the derivative $f'(a)$ is a linear map $T_aM \to T_0\mathbb{R} = \mathbb{R}$, i.e. an element of the conjugate space T_a^*M.

Theorem 3. *$f'(a) = 0$ if and only if $f \in \mathcal{J}_a^2$.*

Proof. Let $f \in \mathcal{J}_a^2$, i.e. $f = \sum_{i=1}^N g_i h_i$ where $g_1, \dots, g_N, h_1, \dots, h_N \in \mathcal{J}_a$. Then $f'(a) = \sum_{i=1}^N (g_i(a)h_i'(a) + g_i'(a)h_i(a)) = 0$.

To prove the converse statement we need the following

Lemma 3 (Hadamard). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $f(0) = 0$ and $f'(0) = 0$. Then there exists a smooth map $A : \mathbb{R}^n \to \mathrm{Symm}(\mathbb{R}^n)$ to the set of symmetric matrices $n \times n$ such that $f(x) = \big(A(x)x, x\big)$ for all $x \in \mathbb{R}^n$, and $A(0) = \frac{1}{2}f''(0)$.*

Proof. One has $f(x) = \int_0^1 \frac{d}{dt} f(tx)\,dt$ (by Newton–Leibnitz) $= \int_0^1 \big(f'(tx), x\big)\,dt$ (by the chain rule) $= \int_0^1 \big(\int_0^t \frac{d}{ds} f'(sx)\,ds, x\big)\,dt$ (Newton–Leibnitz) $= \int_0^1 \big(\int_0^t f''(sx)x\,ds, x\big)\,dt$ (chain rule) $= \big(A(x)x, x\big)$ where $A(x) = \int_0^1 \int_0^t f''(sx)\,dsdt$. Apparently, $A(0) = f''(0) \int_0^1 \int_0^t dsdt = \frac{1}{2} f''(0)$. By the symmetry of mixed derivative the matrix $A(x)$ is symmetric. $\square$

Let $f(a) = 0$ and $f'(a) = 0$. Consider a system of coordinates (U, V, x) such that $a \in U$ and $x(a) = 0$; take $\varphi = f \circ x^{-1} : V \to \mathbb{R}$, so that $\varphi(0) = 0$ and $\varphi'(0) = 0$. By the Hadamard's lemma $\varphi(x) = (A(x)x, x) = \sum_{i,j=1}^n a_{ij}(x)x_i x_j = \sum_{i=1}^n u_i(x)x_i$, where $u_i(x) = \sum_{j=1}^n a_{ij}(x)x_j$. So, $\varphi \in \mathcal{J}_0^2 \subset C^\infty(V)$.

Take now $\varepsilon > 0$ such that $B(0, 2\varepsilon) \subset V$ (a ball or radius 2ε centered at 0), and denote by $\lambda : V \to \mathbb{R}$ a smooth function equal to 1 in the ball $B(0, \varepsilon)$ and equal to 0 outside $B(0, 2\varepsilon)$. Define $\widetilde{\varphi} \stackrel{\text{def}}{=} \varphi\lambda^2$, $\widetilde{\mu_i} = u_i\lambda$ и $\widetilde{\nu_i} = x_i\lambda$. Then $\widetilde{\varphi} = \sum_{i=1}^N \widetilde{\mu_i}\widetilde{\nu_i} \in J_q^2$. On the other side, the function $\Lambda \stackrel{\text{def}}{=} \lambda \circ x : U \to \mathbb{R}$ can be extended by zero to the whole of M keeping it smooth; one can do the same to the functions $g_i \stackrel{\text{def}}{=} \mu_i \circ x$, $h_i \stackrel{\text{def}}{=} \nu_i \circ x$ and $\widetilde{f} \stackrel{\text{def}}{=} \widetilde{\varphi} \circ x$.

Now $\widetilde{f} = \sum_{i=1}^N g_i h_i \in J_a^2$. On the other hand $f - \widetilde{f} = f(1 - \Lambda^2) \in J_a^2$ (because $\Lambda(a) = 1$); therefore, $f \in J_a^2$. $\square$

Corollary 2. *$\mathcal{J}_a / \mathcal{J}_a^2 = T_a^*M$.*

Proof. By Theorem 3 the quotient $\mathcal{J}_a / \mathcal{J}_a^2$ maps injectively to T_a^*M. Let $(x_1, \dots, x_n)$ be coordinates in a chart $U \ni a$, and let Λ be a function defined in the proof of Theorem 3. Then the functions $y_i(b) \stackrel{\text{def}}{=} \Lambda(b)(x_i(m) - x_i(a))$ take zero values outside some neighbourhood of the point a. Therefore, they can be smoothly extended to the whole of M by zero. Obviously, $y_i(a) = 0$ (that is, $y_i \in \mathcal{J}_a$), and $\langle y_i'(a), \frac{\partial}{\partial x_i}(a)\rangle = 1$. It implies the linear independence of the functionals $y_i'(a) : T_aM \to \mathbb{R}$, $i = 1, \dots, n$. So, the dimension of the image of $\mathcal{J}_a / \mathcal{J}_a^2$ under the map to T_a^*M is not less than $n = \dim T_a^*M$ — hence the map is surjective. $\square$

So, a vector $v \in T_aM = (\mathcal{J}_a / \mathcal{J}_a^2)^*$ can be interpreted as a linear functional $v : \mathcal{J}_a \to \mathbb{R}$ such that $\mathcal{J}_a^2 \subset \operatorname{Ker} v$; extend it to the functional $v : C^\infty(M) \to \mathbb{R}$ by setting $v(1) \stackrel{\text{def}}{=} 0$.

Theorem 4. *The functional $v : C^\infty(M) \to \mathbb{R}$ so obtained satisfies the Leibnitz rule: $v(fg)(a) = f(a)v(g) + g(a)v(f)$ for all $f, g \in C^\infty(M)$. Conversely, if $v : C^\infty(M) \to \mathbb{R}$ satisfies the Leibnitz rule then it is an element of T_aM.*

Proof. Suppose $v \in T_aM$. Then $v(fg) = v(fg + f(a)g(a))$ because $v(1) = 0$. Since $f - f(a), g - g(a) \in \mathcal{J}_a$, one has $(f - f(a))(g - g(a)) \in \mathcal{J}_a^2$, hence $0 = v((f - f(a))(g - g(a))) = v(fg + f(a)g(a)) - f(a)v(g) - g(a)v(f)$, so v satisfies the Leibnitz rule.

Conversely, let v satisfy the Leibnitz rule. Then $v(1) = v(1^2) = 1 \cdot v(1) + 1 \cdot v(1) = 2v(1)$, so $v(1) = 0$. If $f, g \in \mathcal{J}_a$ then $v(fg) = f(a)v(g) + g(a)v(f) = 0$, so $\mathcal{J}_a^2 \subset \operatorname{Ker} v$ by linearity. Thus, $v \in (\mathcal{J}_a / \mathcal{J}_a^2)^* = T_aM$. $\square$

Let now Z be a smooth vector field on M. Then for any $f \in C^\infty(M)$ one can define a function $Zf \in C^\infty(M)$ setting $(Zf)(a) \stackrel{\text{def}}{=} Z(a)(f)$ where $Z(a) \in T_aM$.

Theorem 5. *The linear operator $Z : C^\infty(M) \to C^\infty(M)$ so defined is a derivation of the algebra $C^\infty(M)$: $Z(fg) = fZ(g) + gZ(f)$. Conversely, every derivation of the algebra $C^\infty(M)$ is a smooth vector field.*

Proof. The first statement follows from Theorem 4. Conversely, let $Z : C^\infty(M) \to C^\infty(M)$ be a derivation. For any point $a \in M$ define a linear map $Z_a : \mathcal{J}_a \to \mathbb{R}$ by $Z_a(f) \stackrel{\text{def}}{=} Z(f)(a)$. By the second statement of Theorem 4 one has $Z_a \in T_aM$. If $a \in U$ where (U, V, x) is a coordinate system then $Z_a \stackrel{\text{def}}{=} \sum_{i=1}^n p_i(a) \frac{\partial}{\partial x_i}(a)$.

To complete the proof it is necessary to show that the functions p_i are smooth. Take a function $\Lambda \in C^\infty(M)$ like in the proof of Theorem 3 ($\Lambda \equiv 1$ in the vicinity of the point a and $\Lambda \equiv 0$ outside the chart U). Consider the functions $\varphi_i(b) \stackrel{\text{def}}{=} \Lambda(b)x_i(b)$ where $b \in U$, and take $\varphi_i(b) = 0$ for $b \notin U$; apparently, $\varphi_i \in C^\infty(M)$ and $Z(\varphi_i)(b) = p_i(b)$ for $b \in U$. But Z is a map $C^\infty(M) \to C^\infty(M)$, so the function p_i is smooth. $\square$