LECTURE 2

ABSTRACT. Tangent bundle of a manifold. Submanifolds and the implicit function theorem.

1. **Definition of the tangent bundle.** Let M be an n-dimensional manifold, and $a \in M$. Take a coordinate system (U, v, x) such that $a \in U$. Let $\gamma : \mathbb{R} \to M$ be a smooth map with $\gamma(0) = a$; we call such maps *curves.* Two curves γ_1 and γ_2 are called equivalent (at a) if $x(\gamma_1(t)) - x(\gamma_2(t)) = o(t)$ as $t \to 0$; notation: $\gamma_1 \sim_a \gamma_2$.

Lemma 1. (1) *$\gamma_1 \sim_a \gamma_2$ if and only if $\frac{d}{dt}x(\gamma_1(t))\big|_{t=0} = \frac{d}{dt}x(\gamma_2(t))\big|_{t=0}$.*
(2) *The notion of equivalence of curves does not depend on the choice of the coordinate system (U, V, x).*
(3) *$\sim_a$ is an equivalence relation on the set of smooth maps $\gamma : \mathbb{R} \to M$ with $\gamma(0) = a$.*

Proof of the lemma is an easy exercise.

The set of all curves $\gamma : \mathbb{R} \to M$ with $\gamma(0) = a$ is thus split into equivalence classes called *tangent vectors* to M at the point a; the set of tangent vectors is called the *tangent space* to M at a and is denoted T_aM. The union of all T_aM, $a \in M$, is called the *tangent bundle* of M and is denoted TM.

The set TM possesses a structure of a $2n$-dimensional manifold defined as follows: let $A \stackrel{\text{def}}{=} \{(U_\alpha, V_\alpha, x_\alpha)\}$ be an atlas in M; take $\mathcal{U}_\alpha \stackrel{\text{def}}{=} \bigcup_{a \in U_\alpha} T_aM$ and $\mathcal{V}_\alpha = V_\alpha \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. Let $v \in T_aM$ be the equivalence class of a curve γ; then define $y_\alpha(v) \stackrel{\text{def}}{=} (x_\alpha(a), \frac{d}{dt}x(\gamma(t))\big|_{t=0}) \in \mathcal{V}_\alpha$. By Property 1 from Lemma 1 the value $y_\alpha(v)$ does not depend on the choice of $\gamma \in v$.

Theorem 1. *The set $\{(\mathcal{U}_\alpha, \mathcal{V}_\alpha, y_\alpha)\}$ is an atlas on TM denoted $\mathcal{T}_A$. If A and B are equivalent atlases on M then $\mathcal{T}_A$ and $\mathcal{T}_B$ are equivalent.*

Proof. Prove first that $\bigcup_\alpha \mathcal{U}_\alpha = TM$. By Lemma 1 the map $y_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$ is an injection. Prove it is a surjection, that is, for any $q \in \mathbb{R}^n$ there exists a curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = a$ and $\frac{d}{dt}x(\gamma(t))\big|_{t=0} = q$. If $q = 0$ then it is enough to take $\gamma(t) \equiv a$. If $q \neq 0$, let $\ell = \{x(a) + qs \mid s \in \mathbb{R}\} \subset \mathbb{R}^n$ be the straight line through $x(a)$ in the direction of q. The open set $V_\alpha \ni x(a)$ contains a ball $B_r(x(a))$; now take a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$, $\varphi'(0) = 1$ and $|\varphi(t)| < r/\,|q|$ for all t. Then one can take $\gamma(t) \stackrel{\text{def}}{=} x^{-1}(x(a) + q\varphi(t))$.

Let $a \in U_\alpha \cap U_\beta$. Then for t close to 0 one has $\gamma(t) \in U_\alpha \cap U_\beta$, too, and therefore

$$\begin{aligned} y_\beta(v) &= \left(\varphi_{\alpha\beta}(x_\alpha(a)), \frac{d}{dt}(\varphi_{\alpha\beta}(x_\alpha(\gamma(t)))\bigg|_{t=0}\right) = \left(\varphi_{\alpha\beta}(x_\alpha(a)), \varphi'_{\alpha\beta}(x_\alpha(a))\,\frac{d}{dt}x_\alpha(\gamma(t))\bigg|_{t=0}\right) \\ &= \Phi_{\alpha\beta}(y_\alpha(v)), \end{aligned}$$

where $\Phi_{\alpha\beta}(b, q) \stackrel{\text{def}}{=} (\varphi_{\alpha\beta}(b), \varphi'_{\alpha\beta}(b)q)$; here $\varphi'_{\alpha\beta}(b)$ is the $n \times n$-matrix such that its matrix elements are partial derivatives of the components of the vector $\varphi_{\alpha\beta}(b) \in \mathbb{R}^n$ by the components of the vector b. Apparently $\Phi_{\alpha\beta}$ is a smooth map.

The Hausdorff property of $\mathcal{T}_A$ is achieved by the usual addition of coordinate subsystems (check!).

Apparently $\mathcal{T}_{A \cup B} = \mathcal{T}_A \cup \mathcal{T}_B$. So if $A \cup B$ is an atlas on M then $\mathcal{T}_A \cup \mathcal{T}_B$ is an atlas on TM. $\square$

So, one defined a smooth $2n$-dimensional structure on TM that depends only on the smooth structure on M. The map $p : TM \to M$ that sends every vector $v \in T_aM$ to the point a is smooth: if x is the coordinate in a chart $U \ni a$ and y is the corresponding coordinate in the neighbourhood $\mathcal{U} \ni v$ then the map p in coordinates looks as a projection $p_{xy}(b, q) = b$. A smooth map $Z : M \to TM$ which is a right inverse to p: $p \circ Z = \mathrm{id}_M$ (in other words, $Z(a) \in T_aM$ for all $a \in M$) is called a *smooth vector field* on the manifold M.

2. **Tangent space as a vector space.** For any $a \in U_\alpha \subset M$ the map $y_\alpha : T_aM \to \mathbb{R}^n$ is one-to-one; also $y_\beta = \varphi'_{\alpha\beta}(x_\alpha(a)) \circ y_\alpha$ is a composition of y_α with a linear map $\mathbb{R}^n \to \mathbb{R}^n$. This means that T_aM has a stucture of an n-dimensional vector space taken from $\mathbb{R}^n$: if $v_1, v_2 \in T_aM$ then $v_1 + v_2 \stackrel{\text{def}}{=} y_\alpha^{-1}(y_\alpha(v_1) + y_\alpha(v_2))$. The vector $v_1 + v_2$ does not depend on the choice of the coordinate system α: $y_\beta^{-1}(y_\beta(v_1) + y_\beta(v_2)) = y_\alpha^{-1}((\varphi'_{\alpha\beta}(x_\alpha(a)))^{-1}(\varphi'_{\alpha\beta}(x_\alpha(a))(y_\alpha(v_1)) + \varphi'_{\alpha\beta}(x_\alpha(a))(y_\alpha(v_2)))) = y_\alpha^{-1}(y_\alpha(v_1) + y_\alpha(v_2)) = v_1 + v_2$; the same for multiplication of $v \in T_aM$ by a scalar $t \in \mathbb{R}$.

Lemma 2. *Let $a \in U$ where $(U, V, x = (x_1, \dots, x_n))$ is a coordinate system. The equivalence classes of curves $\gamma_i(t) \stackrel{\text{def}}{=} x^{-1}(x_1(a), \dots, x_i(a) + t, \dots, x_n(a))$, $i = 1, \dots, n$, form a basis in T_aM.*

The lemma follows immediately from Statement 1 of Lemma 1: one has then $\frac{d}{dt}(x(\gamma_i(t)))\big|_{t=0} = (0,\ldots,1,\ldots,0)$ (1 in the i-th position); such vectors form a basis in $\mathbb{R}^n$. The basis in T_aM so defined is usually denoted $\frac{\partial}{\partial x_1}(a),\ldots,\frac{\partial}{\partial x_n}(a)$; this notation will be explained later.

Example 1. Let $M = S^1 = \{(x,y) \stackrel{\mathrm{def}}{=} u^2+v^2 = 1\}$. Take an atlas $\{(U_1,(-\pi,\pi),x_1),(U_2,(-\pi,\pi),x_2)\}$ of two coordinate systems where $U_1 = S^1\setminus\{(-1,0)\}$ and $U_2 = S^1\setminus\{(1,0)\}$; $x_1(q) \in (-\pi,\pi)$ is the angle (counterclockwise) between the radius of q and the positive direction of the x axis, and $x_2(q) \in (-\pi,\pi)$ the angle (counterclockwise) between the radius of q and the negative direction of the x axis. The transition map is $\varphi_{12}(s) = s+\pi$ for $-\pi < s < 0$ and $\varphi_{12}(s) = s-\pi$ for $0 < s < \pi$. It is easy to see that the atlas is equivalent to the standard one. Let $\gamma_a(t) \in S^1$ be a point whose radius forms an angle $t \bmod 2\pi$ with the radius of the point a. Then $y_1(\gamma_a) = (x_1(a),1)$ and $y_2(\gamma_a) = (x_2(a),-1)$ for all a. Thus, the curve γ_a represents a nonzero element $v_a = \frac{\partial}{\partial x_1}(a) = \frac{\partial}{\partial x_2}(a) \in T_aM$; since $\dim T_aM = 1$, any other vector $v \in T_aM$ is equal to zv_a for some $z \in \mathbb{R}$. Thus a map sending v to the pair (a,z) is a diffeomorphism between TS^1 and $S^1\times\mathbb{R}$.

3. **Derivative of a smooth map.** Let $f : M_1 \to M_2$ be a smooth map between two manifolds (possibly of different dimensions). Let $\gamma : \mathbb{R} \to M_1$ be a smooth map with $\gamma(0) = a \in M_1$ represent a vector $v \in T_aM_1$; then the smooth map $f\circ\gamma : \mathbb{R} \to M_2$ represents a vector $w \in T_{f(a)}M_2$.

Lemma 3. *The vector w depends only on the vector v, not on a particular choice of the map γ representing v. The mapping $f'(a) : T_aM_1 \to T_{f(a)}M_2$ defined as $f'(a)v \stackrel{\mathrm{def}}{=} w$ is linear. It satisfies the chain rule: if $h = g\circ f$ where $g : M_2 \to M_3$ is a smooth map then $h'(a) = g'(f(a))\circ f'(a)$.*

The mapping $f'(a)$ is called, as the notation suggests, the derivative of the map f at the point a.

The main idea of the proof. Fix coordinate systems x_1 and x_2 in the neighbourhoods $U_1 \ni a$ and $U_2 \ni f(a)$ on M_1 and M_2, respectively. Let the coordinate form of the map f is $f_{12} = x_2\circ f\circ x_1^{-1} : V \to \mathbb{R}^n$ where $V \subset \mathbb{R}^n$ is an open set containing $x(a)$. An easy calculation shows that the coordinate form $y_2\circ f'(a)\circ y_1^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ of the map $f'(a)$ in the corresponding coordinates y_1 and y_2 on TM_1 and TM_2 is equal to $f_{12}'(x(a))$, that is, a linear map with the $n\times n$-matrix made of partial derivatives of the components of $f_{12}(b)$ by the components of b. ☐

4. **Submanifolds and the implicit function theorem.** A subset $N \subset M$ of an m-dimensional manifold M is called a submanifold of dimension $n \le m$ if for any $a \in N$ there is a coordinate system $(U,V,x = (x_1,\ldots,x_m))$ in M with $a \in U$ such that $N\cap U = \{b \in U \mid x_{n+1}(b) = \cdots = x_m(b) = 0\}$. A submanifold has a naturally defined structure of a smooth n-dimensional manifold: the charts are $\tilde{U} = U\cap N$ where U are the charts mentioned above, and the coordinates in $\tilde{U}$ are $x_1,\ldots,x_n$.

Theorem 2 (implicit function theorem for manifolds)**.** *Let $f : M \to K$ be a smooth map between manifolds of dimensions m and $k \le m$, respectively. Let $c \in K$ be a regular value, that is, for any $x \in f^{-1}(c)$ the rank of the linear map $f'(x) : T_xM \to T_cK$ is equal to k. Then $f^{-1}(c)$ is a smooth submanifold of M of dimension $m-k$.*

Proof. Let $f(a) = c$, and $U \ni a$, $W \ni c$ be charts in M and K, respectively. Coordinate maps identify them with open subsets in $\mathbb{R}^m$ and $\mathbb{R}^k$; we will omit these maps in formulas writing simply $U \subset \mathbb{R}^m$, $W \subset \mathbb{R}^k$. Without loss of generality take $c = 0$. By the implicit function theorem for $\mathbb{R}^N$ there exists an open subset $V \subset \mathbb{R}^{m-k}$ and a smooth map $g : W\times V \to U$ such that $f(g(w,v)) = w$ for all $w \in W$, $v \in V$. Now $g^{-1} : g(W\times V) \to W\times V$ is the required coordinate map; $g(W\times V) \subset U$ is a chart. ☐

Example 2. Submanifolds of M of dimension m are open subsets $U \subset M$, and only they.

Example 3. A sphere $S = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_1^2+\cdots+x_n^2 = 1\}$ is $f^{-1}(1)$ where $f : \mathbb{R}^n \to \mathbb{R}$ is given by $f(x) = x_1^2+\cdots+x_n^2$. The derivative $f'(x) = 2x \ne 0$ for $x \in S$, so S is a submanifold of $\mathbb{R}^n$ of dimension $n-1$.

Example 4. $N = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$ is not a 1-dimensional submanifold of $\mathbb{R}^2$, Indeed, let U be a chart containing $(0,0)$. By the implicit function theorem there exists an open ball $V \subset U$ such that the set $\{b \in V \mid x_2(b) = 0\}$ is an image of a smooth embedding $\gamma : \mathbb{R} \to \mathbb{R}^2$ (an embedded curve) and is therefore homeomorphic to the real line. An intersection of N with an open ball is not homeomorphic to a line (it splits into 4 components if a point $(0,0)$ is deleted, while a line splits in two).

The set $N = \{(x,y) \in \mathbb{R}^2 \mid xy = a\}$ for any $a \ne 0$ is a 1-submanifold, as follows from Theorem 2: $f(x,y) = xy$, $f'(x,y) = (y,x) \ne 0$ if $xy = a \ne 0$.

Example 5. A submanifold need not be a closed set (it may be even open) but should be locally closed: for any $a \in N$ there exists an open set $U \subset M$, $a \in U$, such that $N\cap U$ is closed in U (because it is defined by equations with continuous functions $x_{n+1},\ldots,x_m$ in the left-hand side). Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be a 2-dimensional torus; it has a naturally defined structure of a 2-dimensional manifold (provide details!). Take a number $\alpha \notin \mathbb{Q}$; the image of the line $\{(t,\alpha t) \in \mathbb{R}^2\}$ under the natural projection $\mathbb{R}^2 \to \mathbb{T}^2$ is homeomorphic to the line, and the projection restricted to the line is a smooth embedding. Nevertheless, the image of the projection is not locally closed (it is dense in $\mathbb{T}^2$), hence, it is not a 1-submanifold.