18.1. Let G be a locally compact abelian group such that the Fourier transform $\mathscr{F}\colon L^1(G) \to C_0(\widehat{G})$ is onto. Prove that G is finite. (*Hint:* look at $\mathscr{F}^*$.)

18.2. (a) Let A be a commutative Banach algebra, and let I be a closed ideal of A. Show that each nonzero character $I \to \mathbb{C}$ uniquely extends to a character $A \to \mathbb{C}$. Show that the resulting map $\mathrm{Max}(I) \to \mathrm{Max}(A)$ is a homeomorphism onto an open subset of $\mathrm{Max}(A)$.

(b) Let G be a nondiscrete locally compact abelian group. Prove that the embedding $\mathrm{Max}\, L^1(G) \to \mathrm{Max}\, M(G)$ (see (a) above) is not onto.

18.3. Let G be a locally compact abelian group. Show that for every $f \in L^1(G)$, $g \in L^2(G)$ we have $(f * g)\widehat{\ }= \hat{f}\hat{g}$.

18.4. Let A be a commutative Banach $*$-algebra, $\mathrm{Max}_*(A) \subset \mathrm{Max}(A)$ be the set of all maximal modular $*$-ideals of A, and $\hat{A}_* \subset \hat{A}$ be the set of all nonzero $*$-characters of A. Show that

(a) the map $\hat{A}_* \to \mathrm{Max}_*(A)$, $\chi \mapsto \mathrm{Ker}\,\chi$, is a bijection;

(b) $\hat{A}_*$ is closed in $\hat{A}$.

18.5. Find the C^*-envelope of **(a)** $C^n[a,b]$; **(b)** $\mathscr{A}(\bar{\mathbb{D}})$.

18.6. (a) Is $C^n[a,b]$ regular? **(b)** Is $\mathscr{A}(\bar{\mathbb{D}})$ regular?

18.7. Let G be a locally compact group. Recall that the *reduced group C^*-algebra* of G is the C^*-subalgebra $C^*_r(G)$ of $\mathscr{B}(L^2(G))$ generated by the image of the left regular representation $\lambda\colon L^1(G) \to \mathscr{B}(L^2(G))$. Prove that, if G is abelian, then the canonical map $C^*(G) \to C^*_r(G)$ is an isomorphism.

18.8. Let G be a locally compact abelian group. Show that

(a) G is compact iff $\widehat{G}$ is discrete;

(b) G is discrete iff $\widehat{G}$ is compact.

18.9. Let G be a compact abelian group equipped with the normalized Haar measure (i.e., the measure of G is 1). Show that the Plancherel measure on $\widehat{G}$ is the counting measure.

18.10. Let (G_i) be a family of compact abelian groups. Construct a topological isomorphism $(\prod G_i)\widehat{\ }\cong \bigoplus \widehat{G}_i$ (where $\bigoplus \widehat{G}_i$ is equipped with the discrete topology).

18.11. Describe $\widehat{\mathbb{Z}}_p$ explicitly.

18.12. (a) Given $x = \sum_k a_k p^k \in \mathbb{Q}_p$ (see Exercise 7.2), let $\lambda(x) = \sum_{k<0} a_k p^k \in \mathbb{Q}$. Define a character $\chi_x\colon \mathbb{Q}_p \to \mathbb{T}$ by $\chi_x(y) = e^{2\pi i \lambda(xy)}$. Show that χ_x is continuous, and that the map $\mathbb{Q}_p \to \widehat{\mathbb{Q}}_p$, $x \mapsto \chi_x$, is a topological isomorphism.

(b) Let μ denote the Haar measure on $\mathbb{Q}_p$ normalized in such a way that $\mu(\mathbb{Z}_p) = 1$ (see Exercise 7.3). Identify $\widehat{\mathbb{Q}}_p$ with $\mathbb{Q}_p$, as in (a). Find the respective Plancherel measure on $\mathbb{Q}_p$.

18.13. Define covariant functors $G \mapsto M(G)$ and $G \mapsto C_b(\widehat{G})$ from the category of locally compact abelian groups to the category of unital Banach $*$-algebras. Show that the Fourier transform is a natural transformation between them.

18.14. Let G be a locally compact abelian group. Given a closed subgroup $H \subset G$, define the *annihilator* of H by $H^\perp = \{\chi \in \widehat{G} : \chi|_H = 1\}$.

(a) Show that $H^{\perp\perp} = H$ (here $\widehat{\widehat{G}}$ is canonically identified with G).

(b) Construct topological isomorphisms $\widehat{G/H} \cong H^\perp$, $\widehat{H} \cong \widehat{G}/H^\perp$.

(c) A short exact sequence $1 \to G' \xrightarrow{i} G \xrightarrow{p} G'' \to 1$ of locally compact abelian groups is *strictly exact* if i is a topological embedding and p is a quotient map. Show that a sequence of the above form is strictly exact iff the dual sequence is strictly exact.