4.1. As in Exercises 2.8 and 2.9, define the convolution product on $L^1(\mathbb{R})$, show that $L^1(\mathbb{R})$ is a commutative nonunital algebra, and prove that the Fourier transform $\mathscr{F}_{\mathbb{R}}\colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is an algebra homomorphism.

4.2. Suppose that $f \in C^1(\mathbb{R})$ and that $f, f' \in L^1(\mathbb{R})$. Prove that $(f')\widehat{\ }(\lambda) = 2\pi i\lambda \hat{f}(\lambda)$ $(\lambda \in \mathbb{R})$. Deduce that if $f \in C^p(\mathbb{R})$ and $f, f', \ldots, f^{(p)} \in L^1(\mathbb{R})$, then $\hat{f}(\lambda) = o(|\lambda|^{-p})$ as $\lambda \to \infty$.

4.3. Formulate and prove a result similar to Exercise 4.2 for the Fourier transform on $\mathbb{T}$.

4.4. Let $t = \mathbf{1}_{\mathbb{R}}$ denote the identity map on $\mathbb{R}$. Let $f \in L^1(\mathbb{R})$, and suppose that $tf \in L^1(\mathbb{R})$. Show that $\hat{f} \in C^1(\mathbb{R})$, and that $\hat{f}'(\lambda) = -2\pi i(tf)\widehat{\ }(\lambda)$ $(\lambda \in \mathbb{R})$. Deduce that if $f, tf, \ldots, t^p f \in L^1(\mathbb{R})$, then $\hat{f} \in C^p(\mathbb{R})$.

4.5. Formulate and prove a result similar to Exercise 4.4 for the Fourier transform on $\mathbb{Z}$.

4.6. Let $\mathscr{F}\colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$ denote the Fourier transform, and let $\hat{\mathscr{F}} = S\mathscr{F}$, where $(Sf)(t) = f(-t)$ $(t \in \mathbb{R})$.

(a) Show that $\mathscr{F}$ and $\hat{\mathscr{F}}$ map the Schwartz space $\mathscr{S}(\mathbb{R})$ continuously into itself.

(b) Suppose that $T\colon \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is a linear map commuting with $\frac{d}{dt}$ and with the multiplication by the coordinate t. Show that $T = c\mathbf{1}_{\mathscr{S}(\mathbb{R})}$ for some $c \in \mathbb{C}$.

(c) Let $f(t) = e^{-\pi t^2}$ $(t \in \mathbb{R})$. Show that $\hat{f} = f$.

(d) Deduce from (a), (b), (c) that $\mathscr{F}\hat{\mathscr{F}} = \hat{\mathscr{F}}\mathscr{F} = \mathbf{1}_{\mathscr{S}(\mathbb{R})}$ on $\mathscr{S}(\mathbb{R})$. In other words, $\mathscr{F}$ is a topological isomorphism of $\mathscr{S}(\mathbb{R})$ onto itself, and $\mathscr{F}^2 = S$ on $\mathscr{S}(\mathbb{R})$.

4.7. (This is an analog of Exercise 4.6 for $\mathbb{Z}$ and $\mathbb{T}$.) Let $C^\infty_{2\pi}(\mathbb{R})$ denote the space of all smooth 2π-periodic functions on $\mathbb{R}$, and let $j\colon C^\infty(\mathbb{T}) \to C^\infty_{2\pi}(\mathbb{R})$ denote the vector space isomorphism given by $(jf)(t) = f(e^{it})$ $(t \in \mathbb{R})$. Given $f \in C^\infty(\mathbb{T})$, define the derivative $f' \in C^\infty(\mathbb{T})$ of f by $f' = j^{-1}(j(f)')$. The higher derivatives $f^{(k)}$ are defined in an obvious way. We endow $C^\infty(\mathbb{T})$ with the topology generated by the family $\{\|\cdot\|_k : k \in \mathbb{Z}_{\geqslant 0}\}$ of seminorms, where $\|f\|_k = \sup_{z\in\mathbb{T}} |f^{(k)}(z)|$.

We define the space of *rapidly decreasing sequences* by

$$s(\mathbb{Z}) = \Big\{x = (x_n) \in \mathbb{C}^{\mathbb{Z}} : \|x\|_k = \sup_{n\in\mathbb{Z}} |x_n||n|^k < \infty \ \forall k \in \mathbb{Z}_{\geqslant 0}\Big\}$$

and topologize $s(\mathbb{Z})$ by the family $\{\|\cdot\|_k : k \in \mathbb{Z}_{\geqslant 0}\}$ of seminorms. Prove that

(a) $\mathscr{F}_{\mathbb{Z}}$ maps $s(\mathbb{Z})$ continuously into $C^\infty(\mathbb{T})$;

(b) $\mathscr{F}_{\mathbb{T}}$ maps $C^\infty(\mathbb{T})$ continuously into $s(\mathbb{Z})$;

(c) $\mathscr{F}_{\mathbb{T}}\mathscr{F}_{\mathbb{Z}} = S_{\mathbb{Z}}$ and $\mathscr{F}_{\mathbb{Z}}\mathscr{F}_{\mathbb{T}} = S_{\mathbb{T}}$, where $(S_{\mathbb{Z}}f)(n) = f(-n)$ and $(S_{\mathbb{T}}g)(z) = g(z^{-1})$ for every $f \in s(\mathbb{Z})$ and $g \in C^\infty(\mathbb{T})$. As a consequence, $\mathscr{F}_{\mathbb{Z}}$ and $\mathscr{F}_{\mathbb{T}}$ are topological isomorphisms between $s(\mathbb{Z})$ and $C^\infty(\mathbb{T})$.

4.8. Given $\lambda \in \mathbb{R}$, let $\chi_\lambda(t) = e^{-2\pi i\lambda t}$ $(t \in \mathbb{R})$. (Recall that the χ_λ's are precisely the unitary characters of $\mathbb{R}$.) Find the Fourier transforms of χ_λ and of the Dirac δ-function δ_λ.

4.9. Let $s'(\mathbb{Z})$ denote the topological dual of $s(\mathbb{Z})$ (i.e., the space of all continuous linear functionals on $s(\mathbb{Z})$). Show that the map $\varphi \mapsto (\varphi(\delta_n))_{n\in\mathbb{Z}}$ is a vector space isomorphism between $s'(\mathbb{Z})$ and the space of *tempered sequences*

$$\Big\{x = (x_n) \in \mathbb{C}^{\mathbb{Z}} : |x_n||n|^{-k} \text{ is bounded for some } k \in \mathbb{Z}_{\geqslant 0}\Big\}.$$

4.10. Let $\mathscr{D}'(\mathbb{T})$ denote the topological dual of $C^\infty(\mathbb{T})$ (i.e., the space of all continuous linear functionals on $C^\infty(\mathbb{T})$). The elements of $\mathscr{D}'(\mathbb{T})$ are called *distributions* on $\mathbb{T}$. Given $f \in L^1(\mathbb{T})$, define $\varphi_f \in \mathscr{D}'(\mathbb{T})$ by $\varphi_f(g) = \int_{\mathbb{T}} fg\, d\mu$. Show that the map $L^1(\mathbb{T}) \to \mathscr{D}'(\mathbb{T})$, $f \mapsto \varphi_f$, is injective.

4.11. Define the Fourier transforms $\mathscr{F}_{\mathbb{Z}}\colon s'(\mathbb{Z}) \to \mathscr{D}'(\mathbb{T})$ and $\mathscr{F}_{\mathbb{T}}\colon \mathscr{D}'(\mathbb{T}) \to s'(\mathbb{Z})$ to be the maps dual to $\mathscr{F}_{\mathbb{T}}\colon C^\infty(\mathbb{T}) \to s(\mathbb{Z})$ and $\mathscr{F}_{\mathbb{Z}}\colon s(\mathbb{Z}) \to C^\infty(\mathbb{T})$, respectively.
(a) Identify $c_0(\mathbb{Z})$ with a subspace of $s'(\mathbb{Z})$ via Exercise 4.9, and identify $L^1(\mathbb{T})$ with a subspace of $\mathscr{D}'(\mathbb{T})$ via Exercise 4.10. Show that that the Fourier transforms on $s'(\mathbb{Z})$ and on $\mathscr{D}'(\mathbb{T})$ extend the "classical" Fourier transforms $\ell^1(\mathbb{Z}) \to C(\mathbb{T})$ and $L^1(\mathbb{T}) \to c_0(\mathbb{Z})$.
(b) (This is an analog of Exercise 4.8.) Calculate the Fourier transforms of the unitary characters and of the Dirac δ-functions on $\mathbb{Z}$ and on $\mathbb{T}$.
(c) (*the Fourier series in $\mathscr{D}'(\mathbb{T})$*). Show that for each $f \in \mathscr{D}'(\mathbb{T})$ we have $f = \sum_{n\in\mathbb{Z}} \hat{f}(n)\chi_{-n}$, where the series converges in the weak* topology on $\mathscr{D}'(\mathbb{T})$ (i.e., the topology of pointwise convergence on elements of $C^\infty(\mathbb{T})$).

4.12. (a) Define a canonical topology on $C^\infty(\mathbb{T}^2)$ by analogy with $C^\infty(\mathbb{T})$.
(b) Show that the map

$$C^\infty(\mathbb{T}) \otimes C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T}^2), \qquad f \otimes g \mapsto ((z,w) \mapsto f(z)g(w)),$$

is injective and has dense image. From now on, we identify $C^\infty(\mathbb{T}) \otimes C^\infty(\mathbb{T})$ with a dense subspace of $C^\infty(\mathbb{T}^2)$ via the above map.
(c) (*tensor product of distributions*). For each φ, ψ in $\mathscr{D}'(\mathbb{T})$ the element $\varphi \otimes \psi \in \mathscr{D}'(\mathbb{T}) \otimes \mathscr{D}'(\mathbb{T})$ may be viewed as a linear functional on $C^\infty(\mathbb{T}) \otimes C^\infty(\mathbb{T})$. Show that $\varphi \otimes \psi$ uniquely extends to a continuous linear functional on $C^\infty(\mathbb{T}^2)$.
(d) Define $\Delta\colon C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T}^2)$ by $(\Delta f)(z,w) = f(zw)$. For each φ, ψ in $\mathscr{D}'(\mathbb{T})$ define the *convolution* $\varphi * \psi \in \mathscr{D}'(\mathbb{T})$ by

$$\langle \varphi * \psi, f \rangle = \langle \varphi \otimes \psi, \Delta f \rangle \qquad (f \in C^\infty(\mathbb{T})).$$

Show that $(\mathscr{D}'(\mathbb{T}), *)$ is a commutative unital algebra containing $L^1(\mathbb{T})$ and $\mathbb{C}\mathbb{T}$ as subalgebras. In particular, the convolution on $\mathscr{D}'(\mathbb{T})$ agrees with those on $L^1(\mathbb{T})$ and on $\mathbb{C}\mathbb{T}$.
(e) Identify $s'(\mathbb{Z})$ with the space of tempered sequences (see Exercise 4.9). Show that $s'(\mathbb{Z})$ is a unital algebra under pointwise multiplication, and that the Fourier transforms $\mathscr{F}_{\mathbb{Z}}$ and $\mathscr{F}_{\mathbb{T}}$ (see Exercise 4.11) are algebra isomorphisms between $s'(\mathbb{Z})$ and $\mathscr{D}'(\mathbb{T})$.

4.13 (*the Poisson summation formula*)**.** Identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$, and define $a\colon \mathscr{S}(\mathbb{R}) \to C^\infty(\mathbb{T})$ by $(af)(t+\mathbb{Z}) = \sum_{n\in\mathbb{Z}} f(t+n)$. Show that we indeed have $af \in C^\infty(\mathbb{T})$ whenever $f \in \mathscr{S}(\mathbb{R})$, and that the diagram

$$\begin{array}{ccc} \mathscr{S}(\mathbb{R}) & \xrightarrow{\mathscr{F}_{\mathbb{R}}} & \mathscr{S}(\mathbb{R}) \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle \text{restr.}} \\ C^\infty(\mathbb{T}) & \xrightarrow[\mathscr{F}_{\mathbb{T}}]{} & s(\mathbb{Z}) \end{array}$$

commutes. Deduce that for each $f \in \mathscr{S}(\mathbb{R})$ we have

$$\sum_{n\in\mathbb{Z}} f(t+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n t} \qquad (t \in \mathbb{R}).$$