8. FUNDAMENTAL THEOREM OF ALGEBRA.

A vector field on $\mathbb{R}^2$ is another name for a continuous map $F : \mathbb{R}^2 \to \mathbb{R}^2$ (imagine $F(v)$ as a plane vector attached to the point v, but this is just a picture). Let $\gamma : S^1 \to \mathbb{R}^2$ be a continuous curve such that $F(\gamma(x)) \neq 0$ for any $x \in S^1$. The rotation $\mathrm{rot}_\gamma(F)$ of the field F on the curve γ is defined as a degree of the map $R_{F,\gamma} : S^1 \to S^1$ defined as $R_{F,\gamma}(x) = F(\gamma(x))/\,|F(\gamma(x))|$. If F_t and γ_t are homotopies of the curve $\gamma = \gamma_0$ and $F = F_0$ such that $F_t(\gamma_t(x)) \neq 0$ for all $x \in S^1$ and all t, then R_{F_t,γ_t} is a homotopy, so that $\mathrm{rot}_{\gamma_t}(F_t) = const$.

Problem 1. (a) Let F and G be two vector fields such that $F(\gamma(x))$ and $G(\gamma(x))$ do not have opposite directions for any $x \in S^1$. Prove that $\mathrm{rot}_\gamma(F) = \mathrm{rot}_\gamma(G)$. (b) Let F and G be two vector fields such that $|G(\gamma(x))| < |F(\gamma(x))|$ for any $x \in S^1$. Prove that $\mathrm{rot}_\gamma(F + G) = \mathrm{rot}_\gamma(F)$.

Let $P(z) = z^n + a_1 z^{n-1} + \dots + a_n$ be a polynomial; $a_1, \dots, a_n \in \mathbb{C}$. Consider P as a vector field on $\mathbb{R}^2 = \mathbb{C}$. For any $r > 0$ define a curve $\gamma_r : S^1 \to \mathbb{C}$ by $\gamma_r(x) = rx$ where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Let also $Q(z) = z^n$, $C(z) = a_n$. Suppose that $a_n \neq 0$.

Problem 2. (a) Prove that $\mathrm{rot}_{\gamma_r}(C) = 0$ and $\mathrm{rot}_{\gamma_r}(Q) = n$ for any $r > 0$. (b) Prove that for $r > 0$ small enough the vectors $P(\gamma_r(x))$ and $C(\gamma_r(x))$ do not have opposite directions for any $x \in S^1$. (c) Prove that for r large enough one has $|(P - Q)(\gamma_r(x))| < |Q(\gamma_r(x))|$ for any $x \in S^1$. (d) Prove that if $P(z) \neq 0$ for any z such that $r_1 \leq |z| \leq r_2$ then $\mathrm{rot}_{\gamma_{r_1}}(P) = \mathrm{rot}_{\gamma_{r_2}}(P)$. (e) Prove the Fundamental theorem of algebra: there exists $z \in \mathbb{C}$ such that $P(z) = 0$.