6. HOPF BUNDLE AND AROUND.

By $|z|$ we denote an absolute value of the complex number z: $|a+bi| = \sqrt{a^2+b^2}$. It is known that $|zw| = |z|\cdot|w|$. By $\overline{z}$ one denotes the complex conjugate of z: $\overline{a+bi} \stackrel{\text{def}}{=} a-bi$; apparently, $|\overline{z}| = |z|$ and $|z|^2 = z\overline{z}$.

Denote by M the set $\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$.

Problem 1 (Heegaard splitting)**.** Prove that (a) M is homeomorphic to S^3. (b) $M_+ \stackrel{\text{def}}{=} \{(z,w) \in M \mid |z| \geq |w|\}$ and $M_- \stackrel{\text{def}}{=} \{(z,w) \in M \mid |z| \leq |w|\}$ are homeomorphic to a solid torus $D^2 \times S^1$ (where D^2 is a 2-disk), and $M_+ \cap M_-$ is homeomorphic to a 2-torus $S^1 \times S^1$.

Problem 2 (Hopf bundle)**.** (a) Consider a map $f : M \to \mathbb{C}P^1$ given by the formula $f(z,w) = [z:w]$. Prove that for any $a \in \mathbb{C}P^1$ the preimage f^{-1} is homeomorphic to a circle S^1. (b) Prove that the set $\Omega = \{(z,s) \in M \mid s \in [0,1]\}$ is homeomorphic to a disk. (c) Prove that the boundary of Ω is $f^{-1}([1:0])$, and for any $a \neq [1:0]$ the intersection $f^{-1}(a) \cap \Omega$ consists of 1 point.

Problem 3 (SU(2))**.** Prove that the matrices $A_{z,w} \stackrel{\text{def}}{=} \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$ where $(z,w) \in M$ form a group, and $A_{z,w}M = M$ for any $(z,w) \in M$.