5. FUNDAMENTAL GROUP.

Problem 1. (a) Find a fundamental group of the Moebius band M, of its boundary ∂M, and describe the homomorphism $\iota_* : \pi_1(\partial M) \to \pi_1(M)$, where $\iota : \partial M \to M$ is the tautological embedding (every point of ∂M is mapped to itself as a point of M). Is ι_* an embedding? (b) Prove that there is no retraction of M onto ∂M, i.e. no continuous map $f : M \to \partial M$ such that $f(x) = x$ for every $x \in \partial M$.

Problem 2. A sphere S^n is defined as $\{(x_0, x_1, \dots, x_n) \mid x_0^2 + x_1^2 + \dots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$. A real projective space $\mathbb{R}P^n$ is the sphere S^n where every point x is glued together with its opposite $-x$. (a) Prove that $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$ (a group of 2 elements). (b) Let $Y_1 \stackrel{\text{def}}{=} \{(x_0, x_1, x_2) \mid || \, |x_0| \le 1/2\}/(x \sim -x \,\forall x) \subset \mathbb{R}P^2$ and $Y_2 \stackrel{\text{def}}{=} \{(x_0, x_1, x_2) \mid || \, |x_0| \ge 1/2\}/(x \sim -x \,\forall x) \subset \mathbb{R}P^2$. Prove that Y_1 is homeomorphic to the Moebius band, Y_2 is homeomorphic to a disk, and the homeomorphisms send the intersection $Y_1 \cap Y_2$ to the boundary of the band and of the disk. Describe the homeomorphism $\iota_* : \pi_1(Y_1) \to \pi_1(\mathbb{R}P^2)$ where $\iota : Y_1 \subset \mathbb{R}P^2$ is the tautological embedding.

Problem 3. Let X_n be the space of n-tuples of pairwise distinct points $a_1, \dots, a_n \in \mathbb{R}^2$, where the tuples made of the same points with different ordering are identified (e.g., X_2 is $\{(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1 \ne x_2\}/\sim$ where $(x_1, x_2) \sim (x_2, x_1)$ for all x_1, x_2). (a) Prove that X_2 is homotopy equivalent to a circle, hence $\pi_1(X_2) = \mathbb{Z}$. (b) Prove that $\pi_1(X_3)$ is infinite. (c) Prove that $\pi_1(X_3)$ is not commutative: there exists loops γ_1 and γ_2 is X_3 such that the products $\gamma_1 \cdot \gamma_2$ and $\gamma_2 \cdot \gamma_1$ are not homotopic.