1. OPEN AND CLOSED SETS IN SUBSETS OF $\mathbb{R}^n$.

Let $A \subset \mathbb{R}^n$. A subset $X \subset A$ is called open if for any $x \in X$ there is an $\varepsilon > 0$ such that for any $y \in A$ such that $|y - x| < \varepsilon$ one has $y \in X$. A set $X \subset A$ is called closed if $A \setminus X$ is open. A map $f : A \to B$ is called continuous if for any open set $X \subset B$ its full preimage $f^{-1}(X) = \{u \in A \mid f(a) \in X\}$ is open. A map f is called a homeomorphism if it is one-to-one, and both f and f^{-1} are continuous.

Problem 1. Let $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a map defined as $\iota(x_1, \dots, x_n) = (x_1, \dots, x_n, 0)$. Prove that ι is a homeomorphism.

Problem 2. Prove or disprove all the statements like "an open/closed subset of an open/closed set is open/closed".

Problem 3. Prove that (a) a union of any collection of open sets is open; (b) an intersection of a finite collection of open sets is open; (c) an intersection of any collection of closed sets is closed; (d) a union of a finite collection of closed sets is closed. (e) Show that one cannot remove finiteness condition from 3(b) and 3(d).

Problem 4. (a) Prove that a set $X \subset A$ is closed if and only if the limit of any convergent (in A) seequence of points $x_1, x_2, \dots \in X$ also belongs to X. (b) Prove that the map is continuous if and only if for any convergent sequence $a_1, a_2, \dots \in A$ the sequence $f(a_1), f(a_2), \dots \in B$ is also convergent and $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.

Problem 5. Prove or disprove all the statements like "an image/preimage of an open/closed subset under a continuous mapping is open/closed".

Problem 6. A subset $A \subset \mathbb{R}^n$ is called discrete if all its subsets are open in it. Describe continuous maps from and to a discrete set into $\mathbb{R}$. Prove that a set is discrete if and only if every its point a is isolated: for some $\varepsilon > 0$ one has $A \cap \{x \in \mathbb{R}^n \mid |x - a| < \varepsilon\} = \{a\}$.