Basic representation theory: practice problem list

Use the following problems to prepare for the final exam. Please note that the exam covers all topics discussed in the course (starting from September). Thus you may also find the practice problem list for the midterm helpful.

Problem 1. Let $\mathbb{C}^*$ be the multiplicative group of all nonzero complex numbers. Describe all irreducible representations of $\mathbb{C}^*$.

Problem 2. Describe all irreducible representations of $\mathbb{C}^* \times \mathbb{C}^*$.

Recall that E_n is the following representation of $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ in $\mathbb{C}$:

$$E_n(z)(w) = z^n w \quad \forall z \in S^1, \ w \in \mathbb{C}.$$

Problem 3. Describe all irreducible representations of $S^1 \times S^1$.

Problem 4. Is it true that every complex representation of the following group is completely reducible:

(1) $S^1 \times SO(3)$;
(2) $\mathbb{R} \times SU(2)$;
(3) $\mathbb{C}^*$;
(4) $\mathbb{C}^* \times S^1$?

Rigorously justify your answer.

Problem 5. Find the representation $E_n \otimes E_m$.

Problem 6. Find a quaternion q with the following properties:

$$q^2 = -1, \quad \mathrm{Re}(qi) = \mathrm{Re}(qj) = \mathrm{Re}(qk).$$

Recall that $Sp(1)$ denotes the group of all unit quaternions under multiplication (the group $SU(2)$ is naturally isomorphic to $SU(2)$).

Problem 7. Find the image under the homomorphism $\pi : Sp(1) \to SO(3)$ of the rotation around the z-axis by the angle $\pi/3$.

Recall that $\mathcal{H}_n \subset \mathcal{P}_n(\mathbb{R}^3, \mathbb{C})$ is the $SO(3)$-invariant subspace consisting of harmonic polynomials.

Problem 8. Prove that the linear map $L : \mathcal{H}_1 \otimes \mathcal{H}_1 \to \mathcal{P}_2(\mathbb{R}^3, \mathbb{C})$ given by the formula

$$L(f \otimes g) = fg$$

is a morphism of $SO(3)$-representations. Find the dimension of the kernel of this morphism.

Problem 9. Let L be as in the preceding problem. Decompose $L(\mathcal{H}_1 \otimes \mathcal{H}_1)$ into a direct sum of irreducible representations.

Problem 10. Find the dimension of the subspace in $\mathcal{P}_5(\mathbb{R}^3, \mathbb{C})$ spanned by the products $Y_{2,i} Y_{3,j}$, where $i = -2, \ldots, 2$, and $j = -3, \ldots, 3$.

Problem 11. Give an example of a nonzero morphism from $Sym^2(\mathcal{H}_3)$ to $\mathcal{P}_6(\mathbb{R}^3, \mathbb{C})$.

Problem 12. Find the restriction of $\Lambda^2(\mathcal{H}_4)$ to the subgroup $SO(2) \subset SO(3)$ consisting of rotations around the z-axis.

Recall that Φ_1 is the standard representation of $SU(2)$ in $\mathbb{C}^2$, and $\Phi_n = Sym^n(\Phi_1)$. Let $\pi : SU(2) \to SO(3)$ denote the 2-1 homomorphism considered in class.

Problem 13. Prove that $\mathcal{H}_n \circ \pi$ is isomorphic to Φ_{2n}.

Problem 14. Suppose that $U \subset \mathcal{P}_n(\mathbb{R}^3, \mathbb{C})$ is a $SO(3)$-invariant subspace. Can the restriction of the $SO(3)$-representation U to $SO(2)$ be isomorphic to

$$2E_{-2} \oplus E_0 \oplus 2E_2?$$

Rigorously justify your answer.

Problem 15. Give an example of a 5-dimensional representation of $GL_2(\mathbb{C})$.

Problem 16. Let U be the standard representation of $GL_2(\mathbb{C})$ in $\mathbb{C}^2$. Prove or disprove: the representation $Sym^2(U)$ is irreducible.

Problem 17. Find the spherical harmonic $Y_{3,1}$ up to proportionality. Express it as a function of the spherical coordinates ϕ and θ.

Problem 18. Suppose that the spherical harmonics $Y_{m,l}$ are normalized so that $\langle Y_{m,l}, Y_{m,l}\rangle = 1$. Find the integral of the function

$$Y_{5,1}\overline{Y}_{5,2} + Y_{3,2} + 3$$

over the sphere S^2.

Problem 19. Represent the function $\cos^2 \phi$ as a linear combination of spherical harmonics.

Recall that the first three Legendre polynomials are

$$L_0(z) = 1, \quad L_1(z) = z, \quad L_2(z) = \frac{3}{2}z^2 - \frac{1}{2}.$$

Problem 20. Represent the polynomial z^2 as a linear combination of the Legendre polynomials.

Recall that the generating function of a sequence a_n is the sum of the power series

$$a_0 + a_1 t + a_2 t^2 + \dots.$$

Problem 21. Find the generating functions for $L_n'(z)$ and for $L_n(1)$.

Problem 22. Find $L_n(0)$.

For sufficiently small ϕ, let $X_\phi \in SU(2)$ be the element such that $\pi(X_\phi)$ is the rotation by angle ϕ around the x-axis. The elements Y_ϕ and Z_ϕ are defined similarly. Define the operator $\hat{s}_x : \mathbb{C}^2 \to \mathbb{C}^2$ by the formula

$$\hat{s}_x = -i\frac{d}{d\phi}|_{\phi=0}(X_\phi).$$

The operators $\hat{s}_x$, $\hat{s}_y$, $\hat{s}_z$ are called the *spin operators.* They act on the Hermitian space $\mathbb{C}^2$ of states equipped with the standard Hermitian inner product (these states can be thought of as spin states of an electron).

Problem 23. Find the matrices of X_ϕ, Y_ϕ, Z_ϕ, $\hat{s}_x$, $\hat{s}_y$, $\hat{s}_z$.

Recall that the commutator $[\hat{A}, \hat{B}]$ of two operators $\hat{A}$, $\hat{B}$ is defined as $\hat{A}\hat{B} - \hat{B}\hat{A}$.

Problem 24. Express the commutators $[\hat{s}_x, \hat{s}_y]$, $[\hat{s}_x, \hat{s}_z]$ and $[\hat{s}_y, \hat{s}_z]$ through the spin operators.

Problem 25. Find all eigenvalues and eigenvectors of the spin operators.

Recall that an eigenvector of $\hat{s}_z$ is a state with a definite z-component of the spin.

Problem 26. Let ψ be the state of the electron, in which the z-component of the spin is $1/2$ (normalized by $\langle\psi|\psi\rangle = 1$). Find the probability that a measurement of the x-component of the spin will also yield $1/2$. (Recall that this probability is equal to the number $|\langle\phi|\psi\rangle|^2$, where ϕ is an eigenvector of $\hat{s}_x$ with the eigenvalue $1/2$ normalized by $\langle\phi|\phi\rangle = 1$).