Stratification of sparse determinantal varieties

Stratification of determinantal varieties. A *stratification* of a set $S \subset \mathbb{C}^k$ is a sequence of closed subsets $S_0 \subset S_1 \subset \ldots \subset S_k = \mathbb{C}^k$, such that the set $S_i \setminus S_{i-1}$ (which is called the i-dimensional stratum) is empty or homeomorphic to an i-dimensional manifold for every i, and S is the union of some of these strata. The determinantal variety $S_{m,n}$ is the set of all complex $(m \times n)$-matrices of rank smaller than $\min(m, n)$; the name comes from the fact that such matrices are characterised by vanishing of maximal minors. It is a closed subset in the space of all complex $(m \times n)$-matrices, and admits the stratification, whose non-empty strata are the sets of the form $\{\text{matrices of rank } r\}$ for $r = 0, \ldots, \min(m, n)$. Exercise: find the dimension of each stratum.

Sparse determinantal varieties. Choose a set $K \subset \{1, \ldots, m\} \times \{1, \ldots, n\}$ and let M_K be the set of all complex $(m \times n)$-matrices, whose (i, j)-entries vanish for $(i, j) \in K$. This is a vector subspace in the space of complex $(m \times n)$-matrices, and the first (easy) problem is as follows:

describe low codimension strata of the *sparse determinantal variety* $S_{m,n} \cap M_K$.

Example: the space of degenerate square triangular 2×2 matrices is the union of two planes, whose intersection is a line L, therefore the two strata are L and its complement.

Effectively degenerate matrices. A matrix of size $m \times n$ with $m \leqslant n$ is said to be *effectively degenerate*, if its rows $a_1, \ldots, a_m$ admit a linear relation $\sum_{i=1}^m \lambda_i a_i = 0$ with $\lambda_i \neq 0$ for every $i = 1, \ldots, m$. The set $T_{m,n}$ of all effectively degenerate complex $(m \times n)$-matrices is dense in $S_{m,n}$, but not closed itself. Thus, the following problem is different from the first one (and is more complicated):

describe low codimension strata of the set $T_{m,n} \cap M_K$.

Example: the space of effectively degenerate square triangular 2×2 matrices is a plane minus its origin O, therefore the two strata are O and its complement.

Motivation. The Kouchnirenko-Bernstein theorem gives an exact upper bound for the number of common roots of n polynomials on $(\mathbb{C} \setminus 0)^n$ with given Newton polytopes. Considering the collection of these polytopes as a matrix of size $1\times n$, a natural generalization of this question is as follows. Given an $(m \times n)$-matrix A, whose entries are polynomials on $(\mathbb{C} \setminus 0)^k$ with given Newton polytopes, find an exact upper bound for the number of points $x \in (\mathbb{C} \setminus 0)^k$ such that $\operatorname{rk} A(x) \leqslant r$. This problem is solved for the maximal and the minimal possible value of r (i.e. for $\min(m, n)$ and 1 respectively), and the study of other cases leads to the aforementioned problems. For instance, the simplest of the remaining cases $m = n = r + 2$ requires understanding of codimension 3 strata of $T_{m,n} \cap M_K$.

General problem.

Describe stratifications of $S_{m,n} \cap M_K$ and $T_{m,n} \cap M_K$.

It is difficult to estimate how complicated this problem is, until any strata of low codimension are well understood. The solution of this problem might rely upon theory of matroids in order to classify sparse degenerate matrices with respect to combinatorial ways of their degeneration, and require construction of normal forms of such matrices in order to verify that the resulting classes are smooth. It would also be important to check that the resulting stratification is a Whitney stratification, or at least a topological stratification.

Literature: "Determinantal variety", "Matroid" – see Wikipedia and references therein. Normals forms – see Arnold, Varchenko, Gusein-Zade, "Singularities of differentiable maps".