MIDTERM EXAM

General remarks:

- Time for the work is 3 hours. The unsolved problems form your homework for this week.
- The exact rules for grading are not known. It is certainly NOT necessary to solve all the problems to get an 'A'.
- An answer to a combinatorial problem, if required, can usually be given in many ways. E.g. a Catalan number c_n (the number of Dyck paths) is $\frac{1}{n+1}\binom{2n}{n}$ (an "explicit" formula), or $\sum_{k=0}^{n-1} c_k c_{n-1-k}$ (a recursion), or $\sum_{n=0}^{\infty} c_n t^n = (1-\sqrt{1-4t})/(2t)$ (a generating function), etc. There are no universal rules which forms of an answer are acceptable and which are not. Generally, the simpler an answer is, the better is the solution (provided the answer is correct and the proofs are complete, of course). "The n-th Fibonacci number multiplied by $\binom{2n}{n-1}$" is a good simple answer. $\binom{17}{6}$ or $\frac{17!}{11!6!}$ is more explicit than 12376.

Good luck!

Problem 1. Prove that the function $f(x) = \sum_{n=1}^{\infty} x^n/n^2$ is not rational (i.e. not a ratio of two polynomials).

Problem 2. How many are there ways to tile a rectangle $2 \times n$ by rectangles 1×2?

Problem 3. An inversion in a tree with the vertices $1, 2, \dots, n$ is a pair (i, j) of vertices such that $1 < i < j$ and the simple path joining the vertex i with the vertex 1 passes through the vertex j. (a) How many are there trees with n vertices and no inversions? (b) How many are there trees with n vertices and 1 inversion?

Problem 4. Call the number k an order of a permutation $x = [x(1), \dots, x(n)] \in S_n$ if $x^k = 1$ and k is the smallest positive integer with this property. (a) Which is the biggest possible order of a permutation $x \in S_{11}$? (b) How many are there permutations $x \in S_{11}$ of that order?

Problem 5. Prove that the Catalan number c_n is equal to the number of ways to arrange the numbers $1, 2, \dots, 2n$ into a $2 \times n$-rectangle so that each row and each column is increasing.