8. TUTTE POLYNOMIAL.

Tutte polynomial is a way to relate to every graph G a polynomial $T_G(x,y)$ such that:

(1) $T_G(x,y) = 1$ if G has no edges.
(2) If e is an edge which is neither a loop nor an isthmus (an edge whose deletion increases the number of connected components of the graph) then $T_G(x,y) = T_{G\setminus e}(x,y) + T_{G/e}(x,y)$; here $G\setminus e$ is the graph G with the edge e deleted, and G/e is the graph G with the edge e contracted.
(3) If e is an isthmus then $T_G(x,y) = xT_{G/e}(x,y)$.
(4) If e is a loop then $T_G(x,y) = yT_{G\setminus e}(x,y)$.

Theorem 1. *A Tutte polynomial exists and is unique.*

Proof. Existence: fix two quantities, q and v, and define $Z_G(q,v) \stackrel{\text{def}}{=} \sum_A q^{k(A)}v^{\varepsilon(A)}$. Here the sum is taken over all spanning subgraphs $A \subset G$ (i.e. the graph A has the same vertices as G and the edges of A are edges of G, possibly not all); $k(A)$ is the number of connected components of A and $\varepsilon(A)$ is the number of its edges.

Let e be an edge of G. All the subgraphs $A \subset G$ split into two classes: those who do not contain e and those who do. The sum over the subgraphs of the first class is equal to $Z_{G\setminus e}(q,v)$; the sum over the second class is $vZ_{G/e}(q,v)$: contracting the edge e does not change the number of connected components of the graph A. Thus $Z_G(q,v) = Z_{G\setminus e}(q,v) + vZ_{G/e}(q,v)$.

Define now $T_G(x,y) = \frac{1}{(x-1)^{k(G)}(y-1)^n} Z_G((x-1)(y-1), y-1)$. It is an excercise to check that T_G possesses all the required properties of the Tutte polynomial.

Uniqueness: if G contains no edges then $T_G(x,y) = 1$ is unique. Suppose that $T_H(x,y)$ is uniquely defined for graphs H containing no more than m edges, and let G contain $m+1$. Take an edge e; then $T_G(x,y)$ is uniquely expressed via Tutte polynomials of $G\setminus e$ and G/e; both graphs contain m edges. $\square$

Example 1. Let $G_G(\lambda)$ be the number of ways to color vertices of the graph G in λ colors so that two vertices joined by an edge are colored differently. Apparently, $C_G(\lambda) = 0$ if G contains a loop. Let an edge e join vertices i and $j \neq i$. The number of colorings of $G\setminus e$ where the colors of i and j are different is equal to $C_G(\lambda)$; the number of colorings where the colors are the same is $C_{G/e}(\lambda)$; thus $C_G(\lambda) = C_{G\setminus e}(\lambda) - C_{G/e}(\lambda)$.

Number the colors $0,\dots,\lambda-1$ and define an operation on a set of colorings adding $1 \bmod \lambda$ to every color. This splits all the colorings of a graph into groups of λ items in each, and for a fixed vertex v all the colorings in a group have different colors in v. If G is disconnected then it is possible to perform such cyclic shift in every connected component separately. Let now e be an isthmus joining i and j; the number of colorings of G/e is the number of colorings of $G\setminus e$ where i and j have the same colors; it follows from the previous considerations that $C_{G\setminus e}(\lambda) = \lambda C_{G/e}(\lambda)$ and therefore $C_G(\lambda) = (\lambda-1)C_{G/e}(\lambda)$.

Consider now a function $\mathcal{C}_G(\lambda) = (-1)^{n(G)}(-\lambda)^{k(G)}T_G(1-\lambda,0)$ where $n(G)$ is the number of vertices of the graph G and $k(G)$, the number of its connected components. If G contains a loop then $T_G(1-\lambda,0) = 0$ and therefore $\mathcal{C}_G(\lambda) = 0$. If e is neither an isthmus nor a loop then $\mathcal{C}_G(\lambda) = (-1)^{n(G)}(-\lambda)^{k(G)}(T_{G\setminus e}(1-\lambda,0) + T_{G/e}(1-\lambda,0)) = \mathcal{C}_{G\setminus e}(\lambda) - \mathcal{C}_{G/e}(\lambda)$. If e is an isthmus then $\mathcal{C}_G(\lambda) = (-1)^{n(G/e)+1}(-\lambda)^{k(G/e)}(1-\lambda)T_{G/e}(1-\lambda,0) = (\lambda-1)\mathcal{C}_{G/e}(\lambda)$. Also if G contains n vertices no edges then $\mathcal{C}_G(\lambda) = \lambda^n = C_G(\lambda)$, so $\mathcal{C}_G(\lambda) = C_G(\lambda)$ for any G by induction.

Example 2. Fix $0 \le p \le 1$ and let $R_G(p)$ be the probability for the graph G to stay connected if every its edge is deleted with probability $1-p$ and stays intact with probability p (independently for all edges). If e is a loop then $R_G(p) = R_{G\setminus e}(p)$. If e is an isthmus then $R_G(p) = pR_{G/e}(p)$ (if an isthmus is deleted then the graph becomes disconnected). If e is neither an isthmus nor a loop then $R_G(p) = pR_{G/e}(p) + (1-p)R_{G\setminus e}(p)$ (the first term is the conditional probability to preserve connectedness if e is intact, and the second, if e is deleted). Reasoning similar to Example 1 one obtains the equality $R_G(p) = (1-p)^{\varepsilon(G)-n(G)+1}p^{n(G)-1}T_G(1, 1/(1-p))$.