27 lines

Problem 1. Verify that the pencil of planes passing through a given line ℓ on a smooth cubic surface $S \subset \mathbb{P}_3$ contains exactly 5 distinct planes $\pi_1, \pi_2, \ldots, \pi_5$ that intersect S along a triple of lines.

Problem 2. Can a smooth cubic surface $S \subset \mathbb{P}_3$ have a plane section that splits into a smooth conic and its tangent line?

Problem 3. Let $p \in S$ be a singular point of a (singular) cubic surface in $\mathbb{P}_3$. Show that there is at least one (but in general 6) lines laying on S and passing through p.

Problem 4. Fix 6 points $\{p_1, p_2, \ldots, p_6\} \subset \mathbb{P}_2 = \mathbb{P}(V)$ such that any 3 are not collinear and all 6 do not lie on the same conic. Let $W = \{F \in S^3V^* \mid F(p_i) = 0 \text{ for each } i = 1,\, 2,\, \ldots,6\}$ be the space of cubic forms on V that vanish at these 6 points. A map $\mathbb{P}_2 \setminus \{p_1, p_2, \ldots, p_6\} \xrightarrow{\psi} \mathbb{P}(W^*)$ takes $p \notin \{p_1, p_2, \ldots, p_6\}$ to a hyperplane in $\mathbb{P}(W)$ formed by all cubic curves passing through p (besides $\{p_1, p_2, \ldots, p_6\}$). Show that:

a) $\dim W = 4$;

b) $S = \overline{\psi\left(\mathbb{P}_2 \setminus \{p_1, p_2, \ldots, p_6\}\right)} \subset \mathbb{P}_3 = \mathbb{P}(W^*)$ is a cubic surface;

c) find 27 pencils of cubics through $\{p_1, p_2, \ldots, p_6\}$ dual to 27 lines on $S \subset \mathbb{P}(W^*)$.

Problem 5 (Schläflische Doppelsechs). The 'double six line configuration' is constructed as follows. Let $[0]$, $[1]$, … , $[5] \subset \mathbb{P}_3$ be six lines such that $[1]$, … , $[5]$ are mutually skew, $[0]$ intersects all of them, and each of $[1]$, … , $[5]$ does not either touch or lay on the quadric drown through any 3 other. Show that: **a)** $\forall\, i = 1, \ldots, 5\ \exists$ unique line $[i'] \neq [0]$ such that $[i'] \cap [j] \neq \varnothing\ \forall\, j \neq i$;

b) $[i'] \cap [i] = [i'] \cap [j'] = \varnothing$ for all $i = 1, \ldots, 5$ and for all $j \neq i$;

c) each of $[1']$, … , $[5']$ does not either touch or lay on the quadric drown through any 3 other;

d) there exists a unique line $[0']$ that intersects each of $[1']$, … , $[5']$

Hint. Let $[0'_1] \neq [1]$ and $[0'_2] \neq [2]$ be the lines, which intersect all $[1']$, … , $[5']$ except for $[1']$ and $[2']$ respectively; show that they have the same intersection points p_3, p_4, p_5 with $[3']$, $[4']$, $[5']$, which may be recovered geometrically using only the lines $[3]$, $[4]$, $[5]$, $[3']$, $[4']$, $[5']$, and $[0]$.

e*) Show that each double six line configuration lies on a smooth cubic surface and explain how to find the other 15 lines laying on it.

Problem 6. Let $\ell_1, \ell_2 \subset S$ be two skew lines on a smooth cubic surface $S \subset \mathbb{P}_3$. Show that the prescription:

$$p \longmapsto (\ell \cap \ell_1, \ell \cap \ell_2)$$

where $p \in S \setminus (\ell_1 \cup \ell_2)$ and ℓ is a unique line through p meeting the both lines ℓ_i, can be extended to a regular morphism $S \xrightarrow{\varrho} \mathbb{P}_1 \times \mathbb{P}_1 = \ell_1 \times \ell_2$, which contracts 5 lines on S to some points on $\mathbb{P}_1 \times \mathbb{P}_1$.

Problem 7*. Show that any smooth cubic $S \subset \mathbb{P}_3$ can be given in appropriate coordinate system by equation $\varphi_1\varphi_2\varphi_3 + \psi_1\psi_2\psi_3 = 0$, where all φ_i, ψ_j are linear homogeneous forms.