Algebraic manifolds

Problem 1. Prove that a projection of affine hypersurface $V(f) \subset \mathbb{A}^n$ from any point $p \notin V(f)$ onto any hyperplane $H \not\ni p$ is dominant.

Problem 2 (Noether's normalization). Show that any affine hypersurface $V(f) \subset \mathbb{A}^n$ admits finite surjective parallel projection onto hyperplane $\mathbb{A}^{n-1} \subset \mathbb{A}^n$.

Problem 3 (geometric definition of dimension). Show that the dimension of irreducible projective variety $X \subset \mathbb{P}_n$ is equal to the minimal integer k such that there exists a projective subspace $L \subset \mathbb{P}_n$ of dimension $n-k-1$ with $L \cap X = \varnothing$.

Problem 4. Let $X \subset \mathbb{P}_n = \mathbb{P}(V)$ be a projective variety of dimension d. Show that $(n-d)$-dimensional projective subspaces $H \subset \mathbb{P}(V)$ intersecting X in a finite number of points form Zariski open subset of grassmannian[1] $\mathrm{Gr}(n+1-d, V)$.

HINT. Use the projection of the incidence graph $\Gamma = \{(x, H) \in X \times \mathrm{Gr}(n+1-d, V) \mid x \in H\}$ onto X to show that Γ is an irreducible projective variety and to find its dimension; then analyze the second projection $\Gamma \longrightarrow \mathrm{Gr}(n+1-d, V)$.

Problem 5. Let $X \xrightarrow{\varphi} Y$ be a regular morphism of algebraic manifolds. Show that isolated[2] points of fibers $\varphi^{-1}(y)$ draw an open subset of X when y runs through Y.

HINT. Use Chevalley's theorem on semi-continuity from the Lecture Notes.

Problem 6. Show that an image of a regular dominant morphism contains an open dense subset.

Problem 7* (Chevalley's constructivity theorem). Prove that an image of any regular morphism of algebraic varieties is *constructive*, i. e. can be constructed from a finite number of open and closed subsets by a finite number of unions, intersections, and taking complements.

Problem 8 (quadratic transformation). Show that prescription $(t_0 : t_1 : t_2) \longmapsto (t_0^{-1} : t_1^{-1} : t_2^{-1})$ produces rational map $\mathbb{P}_2 \overset{q}{\dashrightarrow} \mathbb{P}_2$ defined everywhere except for 3 points, find these points, clarify how does q act on a triple of lines joining the pairs of these points, and describe $\operatorname{im} q$.

Problem 9 (graph of rational map). Let $X \overset{\psi}{\dashrightarrow} Y$ be a rational map regular on open dense $U \subset X$. Define its *graph* $\Gamma_\psi \subset X \times Y$ as Zariski closure of $\{(x, \psi(x)) \in X \times Y \mid x \in U\}$.

a) Show that a graph of the natural rational map $\mathbb{A}^{n+1} \dashrightarrow \mathbb{P}_n$, which sends $P \in \mathbb{A}^{n+1}$ to $(OP) \in \mathbb{P}_n$, is isomorphic to the blow up of the origin.

b*) Let Γ be the graph of quadratic transformation from Prb. 8. Describe the fibers of both projections of G onto source and target planes.

Problem 10. Find all lines lying on

a) a singular projective cubic surface with affine equation $xyz = 1$.

HINT. Show that there are no lines in the initial affine chart

b*) the Fermat cubic surface $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$

HINT. C_F is preserved by the permutations of the coordinates; up to permutations, a pair of linear equations for $\ell \subset C_F$ can be reduced by the Gauss method to $x_0 = \alpha x_2 + \beta x_3$, $x_1 = \gamma x_2 + \delta x_3$; substitute this in Fermat's cubic equation, show that $\alpha\beta\gamma\delta = 0$ etc

Problem 11. Show that there exists a unique homogeneous polynomial P on the space of homogeneous forms of degree 4 in 4 variables such that P vanishes at f iff the surface $f = 0$ in $\mathbb{P}_3$ contains a line.

HINT. Show that the incidence graph $\Gamma = \{(\ell, S) \in \mathrm{Gr}(2,4) \times \mathbb{P}(S^4(\mathbb{C}^4)^*) \mid \ell \subset S\}$ is a projective variety and use the projection $\Gamma \longrightarrow \mathrm{Gr}(2,4)$ to show that Γ is irreducible and find its dimension; then find a finite non-empty fiber for the second projection $\Gamma \longrightarrow \mathbb{P}(S^4(\mathbb{C}^4)^*)$.

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[1] this grassmannian parameterizes all $(n-d)$-dimensional subspaces of $\mathbb{P}(V)$

[2] a point $p \in M$ is called *isolated* point of a subset $M \subset X$ in a topological space X, if it has an open neighborhood $U \ni p$ such that $U \cap M = \{p\}$