Curves.

Problem 1. For the plane curves listed below find all singular points[1], compute their multiplicities, look how many branches come to each singularity and what are their geometric tangents
a) $(x_0+x_1+x_2)^3 = 27\,x_0x_1x_2$ **b)** $x^2y+x\,y^2 = x^4+y^4$ **c)** $(x^2-y+1)^2 = y^2(x^2+1)$

Problem 2. Over $\mathbb{C}$ find all singular points of plane quartic $(x_0^2+x_1^2)^2+3\,x_0^2x_1x_2+x_1^3x_2=0$ and use projection from singularity onto line to write an explicit rational parametrization of this curve.

Problem 3 (rational normal curves). Show that all the following descriptions define a curve $C\subset\mathbb{P}_n$ that can be identified with the Veronese curve by appropriate projective linear isomorphism[2]:
a) C is image of map $\varphi:\mathbb{P}_1\longrightarrow\mathbb{P}_n$ sending $t=(t_0:t_1)\in\mathbb{P}_1$ to $(f_0(t):f_1(t):\cdots:f_n(t))$, where $f_\nu(t)$ are linearly independent homogeneous polynomials of degree n.
b) C is image of map $\varphi_{p_0,p_1,\dots,p_n}:\mathbb{P}_1\longrightarrow\mathbb{P}_n$ sending $t=(t_0:t_1)\in\mathbb{P}_1$ to

$$\left(\frac{1}{\det(p_0,t)}:\frac{1}{\det(p_1,t)}:\cdots:\frac{1}{\det(p_n,t)}\right),$$

where $p_1,p_2,\dots,p_n\in\mathbb{P}_1$ are pairwise distinct and $\det(p_\nu,t)=\alpha_\nu t_1-\beta_\nu t_0$ for $p_\nu=(\alpha_\nu:\beta_\nu)$.
c) Pick up $n+3$ points $p_1,p_2,\dots,p_n$, a, b, $c\in\mathbb{P}_n$ with no $n+1$ in the same hyperplane and write $\ell_i\simeq\mathbb{P}_1$ for a pencil of hyperplanes passing through $n-1$ points p_ν (all except for p_i). Let $\psi_{ij}:\ell_j\xrightarrow{\sim}\ell_i$ be projective linear isomorphism sending 3 hyperplanes passing through a, b, c in ℓ_j to the similar hyperplanes in ℓ_i. Then C is the incidence graph $C=\bigcup\limits_{H\in\ell_1}H\cap\psi_{21}(H)\cap\ldots\cap\psi_{n1}(H)$.
d) C is an arbitrary curve of degree[3] n that does not lie in a hyperplane.

Problem 4. Show that any $n+3$ points on $\mathbb{P}_n$ with no $n+1$ in the same hyperplane lie on a unique rational normal curve.

Problem 5. Let two collections of $(n+3)$ points $p_0,p_1,\dots,p_{n+2}$, $p'_0,p'_1,\dots,p'_{n+2}\in\mathbb{P}_n$ satisfy the condition of the previous problem. Draw rational normal curves C, C' through them and identify both curves with $\mathbb{P}_1\simeq\ell_1$ using Prb. 3(c) with $(a,b,c)=(p_0,p_{n+1},p_{n+2})$. Write $\widetilde{p}_i,\widetilde{p}'_i\in\mathbb{P}_1$ for the images of the corresponding points under this identification. Show that projective linear isomorphism $\mathbb{P}_n\xrightarrow{\sim}\mathbb{P}_n$ sending p_i to p'_i for all i does exist iff the cross-ratios $[\widetilde{p}_0,\widetilde{p}_1,\widetilde{p}_2,\widetilde{p}_i]=[\widetilde{p}'_0,\widetilde{p}'_1,\widetilde{p}'_2,\widetilde{p}'_i]$ coincide for all $i=3,\,4,\,\dots,\,(n+2)$.

[1] including ones lying at infinity
[2] any such a curve C is called *a rational normal curve*
[3] that is, having n distinct intersection points with generic hyperplane, where «generic» means «running through some Zariski open set in $\mathbb{P}_n^\times$»