Commutative algebra draught

Everywhere below $\Bbbk$ is arbitrary algebraically closed field, $\mathbb{A}^n = \Bbbk^n$ means affine space over $\Bbbk$

Problem 1. Give an example of proper non-principal ideal in **a)** $\Bbbk[x,y]$ **b)** $\mathbb{Z}[x]$.

Problem 2. Given an ideal $I \subset A$ in a commutative ring A, we write $\sqrt{I}$ for its *radical*

$$\sqrt{I} \stackrel{\text{def}}{=} \bigcup_{n \geqslant 1} \{a \in A \,|\, a^n \in I\} \ .$$

Given two ideals $I, J \in \Bbbk[x_1, x_2, \ldots, x_n]$, let $K = \{ab \,|\, a \in I, b \in J\}$ and $IJ \subset A$ be an ideal spanned by K. Construct counter examples or prove the following questionable coincidences:

a) $K = IJ$ is already an ideal as it is; **b)** $K = I \cap J$ (by the way, is $I \cap J$ an ideal?);

c) $IJ = I \cap J$; **d)** $I + J = A \Rightarrow IJ = I \cap J$; **e)** $V(I) \cup V(J) = V(IJ)$;

f) $V(I) \cup V(J) = V(I \cap J)$; **g)** $\sqrt{IJ} = \sqrt{I \cap J}$; **h)** $\sqrt{IJ} = \sqrt{I}\sqrt{J}$;

i) $(I = \sqrt{I} \ \ \& \ \ J = \sqrt{J}) \quad \Rightarrow \quad IJ = \sqrt{IJ}$.

Problem 3*. Which of the following rings are noetherian:

a) a ring of formal power series $A[[t]]$ over a noetherian ring A;

b) a ring of power series $f(z) \in \mathbb{C}[[z]]$ converging everywhere in $\mathbb{C}$;

c) $\{p(z)/q(z) \in \mathbb{C}(z) \,|\, q(z) \neq 0 \text{ for } |z| \leqslant 1\}$;

Problem 4. Let $d \in \mathbb{Z}$. Describe the integer closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{d}] \stackrel{\text{def}}{=} \mathbb{Q}[x]/(x^2 - d)$ for

a) $d = \sqrt[3]{-1} \in \mathbb{C} \smallsetminus \mathbb{R}$; **b)** $d = 3$; **c)** $d = 5$; **d*)** arbitrary $d \in \mathbb{N}$ not divisible by squares.

Problem 5. Let $B \supset A$ be an extension of commutative rings such that B is finitely generated as A-module. Prove that $\mathfrak{m}B \neq B$ for any maximal ideal $\mathfrak{m} \subset A$.

Problem 6*. Let $\mathbb{F}$ be a field finitely generated as $\mathbb{Z}$-algebra[1]. Show that $\mathbb{F}$ is finite as a set.

Problem 7. Let $J = (xy, yz, zx) \subset \Bbbk[x,y,z]$. Describe $V(J) \subset \mathbb{A}^3$ and $I(V(J)) \subset \Bbbk[x,y,z]$. Is it possible to define the same variety by 2 polynomial equations?

Problem 8. Find $f \in I(V(J)) \setminus J$ for $J = (x^2 + y^2 - 1, y - 1) \subset \Bbbk[x,y]$.

Problem 9. Describe $V(J) \subset \mathbb{A}^3$ and $I(V(J)) \subset \Bbbk[x,y,z]$ for:

a) $J = (xy, (x-y)z)$ **b)** $J = (xy + yz + zx, x^2 + y^2 + z^2)$

Problem 10. Let a polynomial f vanish along a hypersurface given in $\mathbb{A}^n$ by a polynomial equation $g = 0$. Prove that each irreducible factor of g divides f.

Problem 11 (Zarisky topology). Let $X \subset \mathbb{A}^n$ be an affine algebraic variety with *coordinate algebra* $\Bbbk[X] \stackrel{\text{def}}{=} \Bbbk[x_1, x_2, \ldots, x_n]/I(X)$. For an ideal $J \subset \Bbbk[X]$, let $V(J) \stackrel{\text{def}}{=} \{p \in X \,|\, f(p) = 0 \ \ \forall\, f \in J\}$ be a subvariety of X defined as the zero set of J. For $f \in \Bbbk[X]$ let $D(f) \stackrel{\text{def}}{=} X \smallsetminus V((f))$.

a) Show that the sets $V(J)$, when J runs through all the ideals of $\Bbbk[X]$, form a system of closed sets for some topology[2] on X, i. e. prove that sets $\varnothing$, X, $\bigcap V(J_\nu)$ (for any family of ideals J_ν), and $V(J_1) \cup V(J_2)$ (for any two ideals $J_{1,2}$) have a form $V(J)$ for appropriate J.

b) Show that each open set is a finite union of *principal open sets* $D(f)$.

c) Show that any open covering of any open $U \subset X$ contains a finite sub-covering.

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[1] w. r. t. action $m \cdot a \stackrel{\text{def}}{=} \underbrace{a + a + \cdots + a}_{m \text{ times}}$

[2] it is called *Zariski topology*