Projective quadrics.

Problem 1. Let $S \subset \mathbb{P}_5 = \mathbb{P}(S^2V^*)$ be the space of singular conics on $\mathbb{P}_2 = \mathbb{P}(V)$. Show that singular points of S correspond to double lines in $\mathbb{P}(V)$ and $\mathrm{Sing}\,(S)$ coincides with an image of the quadratic Veronese embedding $\mathbb{P}(V^*) \xhookrightarrow{v_2} \mathbb{P}_5$. For non singular $q \in S$, which corresponds to split conic $\ell_1 \cup \ell_2 \subset \mathbb{P}(V)$, prove that the tangent space T_qS, for S at q, consists of all conics passing through $\ell_1 \cap \ell_2$.

Problem 2. Let a line $(pq) \subset \mathbb{P}_n$ intersect a smooth quadric $Q \subset \mathbb{P}_n$ in two distinct points r, s. Show that p lies on the polar of q w. r. t. Q iff $[p, q; r, s] = -1$.

Problem 3 (cross ratio on a smooth conic). By the definition, a cross ratio $[a, b, c, d]$, of 4 points on a smooth conic C , is the cross ratio of 4 lines $[(pa), (pb), (pc), (pd)]$ in the pencil of lines passing through some point $p \in C$. Show that it does not depend on a choice of $p \in C$ and $[a, b, c, d] = -1$ iff the pole of line (ab) w. r. t. C lies on line (cd). (In this case the chords $[ab]$ and $[cd]$, of C, are called *conjugated*.)

Problem 4. Let the vertexes of quadrangle a, b, c, d lie on a smooth conic $C \in \mathbb{P}_2$. Show that the triangle xyz (whose vertices are the intersection points of opposite sides of the quadrangle) is autopolar w. r. t. C (that is, the pole of each vertex is the line containing the opposite side).

Problem 5 (Desargues's theorem). Show that if two triangles correspond to each other in such a way that the joins of their corresponding vertices are concurrent[1], then the intersections of their corresponding sides are collinear, and conversely (such a pair of triangles is called *perspective*).

Problem 6. Show that two triangles on projective plain are perspective iff they are polar to each other w. r. t. some smooth conic.

Problem 7. How many lines cross each of 4 given pairwise skew lines in:
a) $\mathbb{CP}_3$ **b)** $\mathbb{RP}_3$ **c*)** $\mathbb{C}^3$ **d*)** $\mathbb{R}^3$
Find all possible answers and indicate those that are stable under small perturbations.

Problem 8. Show that tangent planes to non-singular quadric $Q \subset \mathbb{P}_n$ form a quadric $Q^\times \subset \mathbb{P}_n^\times$ in the dual space and express its Gram matrix in terms of the Gram matrix of Q in the dual basis.

Problem 9 (generalization of the previous problem). Let A be non singular and B be an arbitrary quadric in $\mathbb{P}_n$. Show that polar hyperplanes of the points of B w. r. t. smooth quadric A form a quadric in the dual space $\mathbb{P}_n^\times$ and express its gram matrix through the Gram matrices of A and B in the dual basis.

Problem 10. Let U, V be 2-dimensional vector spaces and

$$Q \simeq \mathbb{P}(U^*) \times \mathbb{P}(V) \subset \mathbb{P}(U^* \otimes V)$$

be the Segre quadric formed by rank 1 linear operators $U \xrightarrow{\xi \otimes v} V$ considered up to proportionality. Show that the tangent plane $T_{\xi\otimes v}Q$ to Q at a point $\xi \otimes v \in Q$ is formed by all linear operators $U \longrightarrow V$ that send 1-dimensional subspace $\mathrm{Ann}\,(\xi) = \{u \in U \mid \xi(u) = 0\}$ into 1-dimensional subspace spanned by v.

[1] i. e. have common intersection point