Linear automorphisms of projective line, conics, and polarities.

Problem 1 (cross ratio). Let a cross ratio of 4 points $p_1,\, p_2,\, p_3,\, p_4$ on a complex projective line

$$[p_1,\, p_2,\, p_3,\, p_4] \overset{\text{def}}{=} (\det(p_1,p_3)\cdot\det(p_2,p_4)) : (\det(p_1,p_4)\cdot\det(p_2,p_3)) = \vartheta \in \mathbb{C}\,.$$

a) Compute $[p_{\sigma(1)},\, p_{\sigma(2)};\, p_{\sigma(3)},\, p_{\sigma(4)}]$ for all 24 permutations $\sigma\in S_4$.
b) Find all $\vartheta\in\mathbb{C}$ such that the number of distinct values obtained from ϑ by permutations of points is less than for a generic choice of ϑ.
c) **(harmonic pairs of points)** Show that $[p_1,\, p_2,\, p_3,\, p_4] = -1$ iff $[p_1,\, p_2,\, p_3,\, p_4] = [p_2,\, p_1,\, p_3,\, p_4]$ and this is a symmetric relation between non-ordered pairs of points[1] (p_1,p_2) and (p_3,p_4) .

Problem 2 (complete quadrangle). Let any 3 of 4 points $a,b,c,d\in\mathbb{P}_2$ be non collinear and write x,y,z for the intersection points of the opposite sides of *quadrangle abcd* (see fig.1◇1). Using Problem 1c) and appropriate central projections on lines, show that the opposite sides of quadrangle crossing at x are harmonic to the sides of triangle xyz in the pencil of lines through x (the same holds for y, z as well).

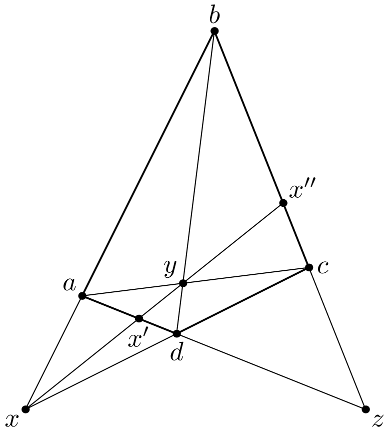

Fig. 1◇1. Complete quadrangle.

Problem 3 (cross axis). Given a linear projective isomorphism $\varphi:\ell_1 \xrightarrow{\sim} \ell_2$ between two lines $\ell_1,\ell_2\subset\mathbb{P}_2$, describe the set of intersections of lines $(x,\varphi(y))\cap(y,\varphi(x))$ when $x\neq y$ are running through ℓ_1. Use this to construct (by means of the ruler only) $\varphi(x)$ for a given $x\in\ell_1$, if the images $\varphi(a),\varphi(b),\varphi(c)$ of some points $a,b,c\in\ell_1$ are known.

Problem 4 (projective automorphisms of a conic). Let us fix some rational parametrization $\mathbb{P}_1 \xrightarrow{\sim} Q$ of non singular conic $Q\subset\mathbb{P}_2$ and say that bijection $Q\xrightarrow{\gamma} Q$ is *linear fractional automorphism* of Q, if it is induced by some linear fractional automorphism of $\mathbb{P}_1$. Show that for any linear fractional automorphism $Q\xrightarrow{\gamma} Q$ there exist two points $p_1,p_2\in Q$ and a line $\ell\subset\mathbb{P}_2$ such that $x\overset{\gamma}{\longmapsto} y$ iff $\pi_\ell^{p_1}x = \pi_\ell^{p_2}y$. Describe the fixed points of γ in terms of ℓ, p_1, p_2 and Q.

Problem 5. Under the conditions of the previous problem assume that linear fractional automorphism $Q\xrightarrow{\gamma} Q$ is given by its action on three distinct points $a,b,c,\in Q$. Using only the ruler, draw some p_1,p_2,ℓ as above and find the fixed points of γ.

Problem 6. Using only the ruler, draw a triangle inscribed in a given non singular conic Q and such that his sides a, b, c pass through 3 given points A, B, C. How many solutions may have this problem?

Hint. Start 'naive' drawing from any $p\in Q$ and denote by $\gamma(p)$ your return point after passing trough A,B,C. Is $p\longmapsto\gamma(p)$ a projective isomorphism of kind described in Problem 4?

Problem 7. Formulate and solve projectively dual problem to the previous one.

Problem 8 (Euclidean polarities). Consider a circle in the real Euclidean affine plane. Using only ruler and compasses, draw **a)** the polar line of a given point[2] **b)** the pole of a given line[3].
c) Describe geometrically the polar map w. r. t. 'imaginary circle' $x^2+y^2=-1$.

Problem 9. Show that all conics passing through points $a=(1:0:0)$, $b=(0:1:0)$, $c=(0:0:1)$, $d=(1:1:1)$ form a line in the space of all conics. Write an explicit equation[4] for these conic family and find all singular conics inside it.

[1] pairs of points satisfying this relation are called *harmonic* to each other
[2] this is especially interesting, when the point is inside the circle
[3] this is especially interesting, when the line does not intersect the circle
[4] it should be a quadratic form whose coefficients depend linearly on two homogeneous parameters