# Projective spaces

**Problem 1.** Let $V$ be an $n$-dimensional vector space over a finite field $\mathbb{F}_q$ of $q$ elements. How many **a)** basises **b)** $k$-dimensional subspaces are there in $V$? **c)** How many points are there in $\mathbb{P}(V)$?

**Problem 2*.** Let $G_n^k(q)$ be a rational function in $q$ that computes the answers to prb. 1b (its value at $q$ is equal to the number of $k$-dimensional vector subspaces in $n$-dimensional vector space over a finite field of $q$ elements as soon as such a field exists). Compute $\lim\limits_{q\to 1} G_n^k(q)$.

**Problem 3.** Consider projective closures of affine curves
**a)** $y = x^2$ **b)** $y = x^3$ **c)** $y^2 + (x-1)^2 = 1$ **d)** $y^2 = x^2(x+1)$
Write down their homogeneous equations and their affine equations in two other standard affine charts on $\mathbb{P}_2$. Try to draw all these affine curves.

**Problem 4 (Pythagorian triples).** Consider $\mathbb{P}_2$ with homogeneous coordinates $(t_0 : t_1 : t_2)$. Let $\ell \subset \mathbb{P}_2$ be the line $t_2 = 0$, $Q \subset \mathbb{P}_2$ be the conic $t_0^2 + t_1^2 = t_2^2$, and $O = (1 : 0 : 1) \in Q$. For each $P = (p : q : 0) \in \ell$ find coordinates of the intersection point $Q \cap (OP)$ different from $O$ and show that the projection from $O$ maps $Q$ bijectively onto $\ell$. Find some polynomials $a(p,q)$, $b(p,q)$, $c(p,q)$ whose values on $\mathbb{Z}\times\mathbb{Z}$ give, up to a common factor, all integer Pythagorian triples $a^2 + b^2 = c^2$ (and only such the triples).

**Problem 5.** Let the real Euclidian plane $\mathbb{R}^2$ be included in $\mathbb{CP}_2$ as a real part of standard affine chart $U_0 \simeq \mathbb{C}^2 \supset \mathbb{R}^2$. **a)** Find two points of $\mathbb{CP}_2$ such that any Euclidean circle will contain them after complexification and projective closuring. **b)** Let degree 2 curve $C \subset \mathbb{CP}_2$ pass through two points from (a) and have at least 3 non collinear points inside the initial $\mathbb{R}^2$. Show that $C \cap \mathbb{R}^2$ is a circle.

**Problem 6 (Veronese map).** Let $S^dV^*$ be the space of all homogeneous degree $d$ polynomials on $n$-dimensional vector space $V$. The Veronese mapping $V^* \xrightarrow{v_d} S^dV^*$ takes a linear form $\psi \in V^*$ to its $d$-th power $\psi^d \in S^dV^*$. Find $\dim S^dV^*$. Does the image of $v_d$ lie in some hyperplane?

**Problem 7 (projecting twisted cubic).** Under the conditions of the previous problem, let $\dim V = 2$ and let us treat $V^*$ as the space of linear forms in two variables $(t_0, t_1)$; then $S^3V^*$ is the space of cubic forms in $(t_0, t_1)$. In this case the image of the Veronese mapping

$$\mathbb{P}_1 = \mathbb{P}(V^*) \xhookrightarrow{\psi\mapsto\psi^3} \mathbb{P}_3 = \mathbb{P}(S^3V^*)$$

is called *a twisted cubic* and is denoted by $C_3 \subset \mathbb{P}_3$. Describe projection of $C_3$:
**a)** from point $t_0^3$ to plane spanned by $3\,t_0^2t_1$, $3\,t_0t_1^2$, and $t_1^3$
**b)** from point $3\,t_0^2t_1$ to plane spanned by $t_0^3$, $3\,t_0t_1^2$, and $t_1^3$
**c)** from point $t_0^3 + t_1^3$ to plane spanned by $t_0^3$, $3\,t_0^2t_1$, and $3\,t_0t_1^2$
More precisely, write an explicit parametric representation for the resulting curve in appropriate homogeneous coordinates in the target plane, then find affine and homogeneous equations for this curve. In each case, find degree of the target curve and draw it in several affine charts covering the whole curve. Are there selfintersections and/or cusps on the target curve (over $\mathbb{C}$)?

**Problem 8.** Let $f : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V)$ be a projective linear isomorphism induced by some linear isomorphism $\widehat{f} : V \xrightarrow{\sim} V$, $\dim V = n+1$. Assume that all fixed points of $f$ are isolated. Estimate a number of them.

**Problem 9.** For $v = (v_0, v_1)$, $w = (w_0, w_1) \in \mathbb{k}^2$ let $\det(v,w) \overset{\text{def}}{=} v_0w_1 - v_1w_0$. Show that for any 4 points $p_1$, $p_2$, $p_3$, $p_4 \in \mathbb{P}_1 = \mathbb{P}(\mathbb{k}^2)$ their *cross-ratio* $[p_1, p_2, p_3, p_4] \overset{\text{def}}{=} \frac{\det(p_1,p_2)\cdot\det(p_3,p_4)}{\det(p_1,p_4)\cdot\det(p_3,p_2)}$ is well defined and does not depend on the choice of homogeneous coordinates. Prove that two quadruples of points can be mapped to each other by some projective linear isomorphismof $\mathbb{P}_1$ iff their cross-ratios coincide.