TRAINING MIDDLE TERM TEST

Problems can be solved in any ordering. Complete solution of each problem contributes 25 points to the final result.

Problem 1. Could the intersection of two smooth conics on projective plane over $\mathbb{C}$ consist of precisely **a)** one **b)** two **c)** three points? (If yes, give an explicit example, if not, explain why).

Problem 2. Let $S \subset \mathbb{P}(S^2V^*)$ be the hypersurface formed by all singular quadrics on $\mathbb{P}_n = \mathbb{P}(V)$. Show that $q \in S$ is a smooth point of S iff the corresponding quadric $V(q) \subset \mathbb{P}_n$ has just one singular point and in this case the tangent space T_qS, to S at q, consists of all quadrics passing in $\mathbb{P}_n$ through the singular point of $V(q)$.

Problem 3. Fix 6 points $\{p_1, p_2, \dots, p_6\} \subset \mathbb{P}_2 = \mathbb{P}(V)$ such that any 3 of them are not collinear and all 6 do not lie on a conic. Write $\mathbb{P}(W)$ for the space of all cubic curves passing through $p_1, p_2, \dots, p_6$. Thus, $W = \{F \in S^3V^* \mid F(p_i) = 0 \text{ for each } i = 1,\, 2,\, \dots,\, 6\}$. For any point $p \in \mathbb{P}_2 \smallsetminus \{p_1, p_2, \dots, p_6\}$ all cubic curves $C \in \mathbb{P}(W)$ passing through p form a hyperplane in $\mathbb{P}(W)$. Algebraically, this hyperplane is the zero set of the linear form $\mathrm{ev}_p : F \longmapsto F(p)$ on W. Sending p to this hyperplane, we get a map

$$\mathbb{P}_2 \smallsetminus \{p_1, p_2, \dots, p_6\} \xrightarrow{\psi} \mathbb{P}(W^*)$$

from $\mathbb{P}_2$ without 6 given points to the dual space for the space of cubics passing through them. Show that **a)** $\dim W = 4$ **b)** the closure $\overline{(\mathrm{im}\,(\psi))}$ is a cubic surface in $\mathbb{P}_3 = \mathbb{P}(W^*)$

Problem 4. Let a finite group G act on a commutative ring A by the ring automorphisms

$$g : A \xrightarrow{\sim} A\,, \quad g \in G\,.$$

Show that A is integral over the subring of G-invariants $A^G = \{a \in A \mid ga = a \ \forall\, g \in G\,\}$.

Problem 5. Let $\Bbbk$ be an arbitrary field and $\Bbbk$-algebra A be finite dimensional as a vector space over $\Bbbk$. Show that the set of proper prime ideals[1] in A is finite and all these ideals are maximal.

[1] recall that ideal $\mathfrak{p} \subset A$ is called *prime* if factor ring $A/\mathfrak{p}$ has no zero divisors