§1 Projective spaces

1.1 Polynomials. Let V be n-dimensional vector space over an arbitrary field $\Bbbk$. Its *dual space* V^* is the space of all $\Bbbk$-linear maps $V \longrightarrow \Bbbk$. Given a basis $e_1, e_2, \dots, e_n$ for V, its *dual basis* for V^* consists of the coordinate forms $x_1, x_2, \dots, x_n$ defined by prescriptions

$$x_i(e_j) = \begin{cases} 1\ , \text{ if } i = j \\ 0\ , \text{ otherwise}\,. \end{cases}$$

One can treat each polynomial $f \in \Bbbk[x_1, x_2, \dots, x_n]$ as a function on V whose value at a vector $v \in V$ with coordinates $(v_1, v_2, \dots, v_n)$ w.r.t. the basis $(e_1, e_2, \dots, e_n)$ is equal to $f(v_1, v_2, \dots, v_n)$, i.e. to the result of substitution of values $v_i \in \Bbbk$ instead of the variables x_i. This gives $\Bbbk$-algebra homomorphism

$$\varphi\ :\ \Bbbk[x_1, x_2, \dots, x_n] \longrightarrow \{\Bbbk\text{-valued functions on } V\}\ . \tag{1-1}$$

LEMMA 1.1
The homomorphism (1-1) is injective[1] iff $\Bbbk$ is infinite.

PROOF. If $\Bbbk$ is finite and consists of q elements, then the space of $\Bbbk$-valued functions on V consists of q^{q^n} elements and is finite as well. Since the polynomial algebra is infinite, $\ker\varphi \neq 0$. The inverse argumentation uses the induction on $n = \dim V$. When $n = 1$, any non zero polynomial $f(x)$ has no more than $\deg f$ roots. Thus, $f(x) \equiv 0$ as soon $f(v) = 0$ for infinitely many $v \in V \simeq \Bbbk$. For $n > 1$ we can write a polynomial f as a polynomial in x_n with the coefficients in $\Bbbk[x_1, x_2, \dots, x_{n-1}]$: $f(x_1, x_2, \dots, x_n) = \sum\limits_{\nu} f_\nu(x_1, x_2, \dots, x_{n-1}) \cdot x_n^\nu$. Evaluating all f_ν at an arbitrary point $(v_1, v_2, \dots, v_{n-1}) \in \Bbbk^{n-1}$, we get a polynomial in x_n with constant coefficients and identically zero values. It should be the zero polynomial by the above reason. Hence, each f_ν gives the zero function on $\Bbbk^{n-1}$. By the inductive assumption, all $f_\nu = 0$ as the polynomials. □

1.2 Spaces. There are two geometric spaces associated with with a given vector space V over a field $\Bbbk$: affine space $\mathbb{A}(V)$ that has the same dimension $\dim \mathbb{A}(V) = \dim V$ and *projective space* $\mathbb{P}(V)$ whose dimension is one less $\dim \mathbb{P}(V) = \dim V - 1$.

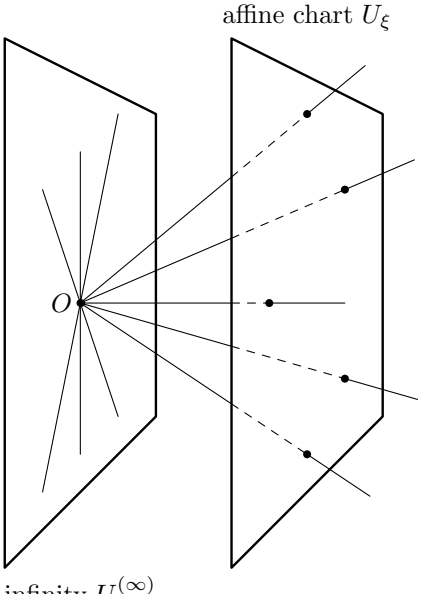

Fig. 1◇1. The projective world.

By the definition, the points of n-dimensional *affine space* $\mathbb{A}^n = \mathbb{A}(V)$ associated with n-dimensional vector space V are the vectors of V. The point corresponding to the zero vector is called *the origin* and is denoted it by O. All other points can be imagined as « the ends» of non zero vectors « drawn» from the origin. The homomorphism (1-1) allows to treat the polynomials as the functions on $\mathbb{A}(V)$. Although this construction does depend on the choice of a basis in V, the resulting space of functions on $\mathbb{A}(V)$, i.e. the image of homomorphism (1-1), does not. It is called an algebra of *polynomial* (or *algebraic*) functions on $\mathbb{A}(V)$. A subset $X \subset \mathbb{A}(V)$ is called *an affine algebraic variety*, if it can be defined by some (maybe infinite) system of polynomial equations.

By the definition, the points of n-dimensional *projective space* $\mathbb{P}_n = \mathbb{P}(V)$, which is associated with a vector space V of dimension $(n+1)$, are 1-dimensional vector subspaces in V or, equivalently, the lines in $\mathbb{A}^{n+1} = \mathbb{A}(V)$ passing through the origin $O \in \mathbb{A}(V)$.

To see them as « usual» points one should use a screen, i.e. some affine hyperplane of codimension 1 that does not contain the origin like on fig. 1◇1. Such a hyperplane $U_\xi \subset \mathbb{A}(V)$ is given by linear equation $\xi(v) = 1$, where $\xi \in V^*$ is a

[1] i. e. different polynomials always give the different polynomial functions on V

non zero linear form on V uniquely defined by the affine hyperplane. Any such a screen is called *an affine chart* on $\mathbb{P}(V)$.

Of course, no affine chart does cover the whole of $\mathbb{P}(V)$. The difference $U_\xi^{(\infty)} \stackrel{\text{def}}{=} \mathbb{P}_n \setminus U_\xi$ consists of all one dimensional vector subspaces lying in n-dimensional vector subspace

$$\operatorname{Ann}\xi = \{v \in V \mid \xi(v) = v\},$$

which is a parallel copy of U_ξ drawn through O. Thus, $U_\xi^{(\infty)} = \mathbb{P}(\operatorname{Ann}\xi)$ is naturally identified with $\mathbb{P}_{n-1}$ and we can represent $\mathbb{P}_n$ as disjoint union $\mathbb{P}_n = \mathbb{A}^n \sqcup \mathbb{P}_{n-1}$. Iterating this procedure, we split $\mathbb{P}_n$ into disjoint union of affine spaces:

$$\mathbb{P}_n = \mathbb{A}^n \sqcup \mathbb{P}_{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{P}_{n-2} = \cdots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \ldots \sqcup \mathbb{A}^0$$

(note that 0-dimensional projective space $\mathbb{P}_0 = \mathbb{P}(\Bbbk^1)$ as well as 0-dimensional affine space $\mathbb{A}^0 = \mathbb{A}(0)$ is the one point set).

1.3 Homogeneous coordinates. Let us fix a basis for V and use it to write vectors $x \in V$ as coordinate rows $x = (x_0, x_1, \ldots, x_n)$. Two vectors $x, y \in V$ represent the same point $p \in \mathbb{P}(V)$ iff they are proportional, i.e. $x_\nu = \lambda\, y_\nu$ for all $\nu = 0, 1, \ldots, n$ for some non zero $\lambda \in \Bbbk$. Thus, the point $p \in \mathbb{P}(V)$ can be coordinated by the collection of n ratios $(x_0 : x_1 : \ \ldots : x_n)$. These ratios are called *homogeneous coordinates* on $\mathbb{P}(V)$ w.r.t. the chosen basis of V.

1.3.1 Projective hypersurfaces. In a contrast with affine geometry, non constant polynomial $f \in \Bbbk[x_0, x_1, \ldots, x_n]$ does not produce a function on $\mathbb{P}(V)$, because its value is changed under rescalling: $f(x) \neq f(\lambda x)$ and is not well defined at $p = (x_0 : x_1 : \ \ldots : x_n) \in \mathbb{P}_n$. Nevertheless, the zero set

$$V(f) \stackrel{\text{def}}{=} \{\, v \in V \mid f(v) = 0 \,\}$$

of any homogeneous polynomial f of arbitrary degree $d \geqslant 1$ is a well defined figure in $\mathbb{P}(V)$, because $f(x) = 0 \iff f(\lambda x) = \lambda^d f(x) = 0$. This figure is called *a projective hypersurface* of degree d. Hypersurfaces of degree 2 are called *quadrics*. Traditionally, quadrics on projective plane are called *conics*.

We write $S^d(V^*) \subset \Bbbk[x_0, x_1, \ldots, x_n]$ for the subspace of all homogeneous polynomials of degree d. Since proportional equations define the same figures, each projective hypersurface of degree d can be depicted by a point in projective space $\mathbb{P}_N = \mathbb{P}(S^d V^*)$ of dimension $N = \binom{n+d}{d} - 1$. This projective space is called *the space of degree d hypersurfaces* in $\mathbb{P}_n = \mathbb{P}(V)$. Traditionally, lines, planes, and 3D-subspaces in the space of hypersurfaces are called, respectively, *pencils*, *nets*, and *webs* of hypersurfaces. An intersection of projective hypersurfaces is called *projective algebraic variety*.

1.4 Local affine coordinates. Besides homogeneous coordinates, which are globally defined on the whole of $\mathbb{P}_n = \mathbb{P}(V)$, one can introduce *local affine coordinates* inside any chosen affine chart $U_\xi \subset \mathbb{P}(V)$. Namely, fix any collection of n linear forms $\xi_1, \xi_2, \ldots, \xi_n \in V^*$ such that $n+1$ forms $\xi, \xi_1, \xi_2, \ldots, \xi_n$ form a basis of V^* and use their values $\xi_i(v)$ at vector $v \in U_\xi$ (rescalled in such a way that $\xi(v) = 1$) as coordinates of the point represented by v.

Note, that local affine coordinates are not linear but *linear fractional* functions of homogeneous coordinates. Indeed, to compute their values at a point

$$p = (x_0 : x_1 : \ \ldots : x_n)$$

one should pass to $v = p/\xi(p)$ in order to have $\xi(v) = 1$. Thus, i-th local affine coordinate of p equals $\xi_i(x)/\xi(x)$. This agrees with the fact that p is visible in chart U_ξ iff $\xi(x) \neq 0$ and sentence « p is running away from U_α to infinity» exactly means that « $\xi(x) \to 0$», which leads to unbounded increasing of the local affine coordinates.

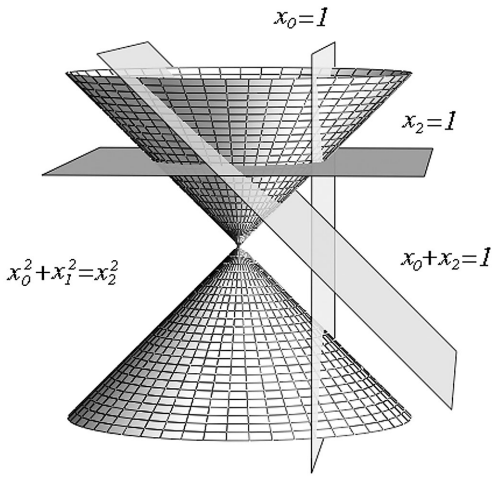

Fig. 1◇2. Conic in charts.

1.4.1 Example: affine conics. Consider local equations of plane conic

$$x_0^2 + x_1^2 = x_2^2 \tag{1-2}$$

in some affine charts. In the chart U_{x_0}, given by the equation $\{x_0 = 1\}$, we can chose local affine coordinates $t_1 = x_1|_{U_0} = x_1/x_0$ and $t_2 = x_2|_{U_{x_0}} = x_2/x_0$. Dividing the both sides of (1-2) by x_2^2, we get for $C \cap U_0$ the equation $t_2^2 - t_1^2 = 1$, i.e. $C \cap U_{x_0}$ is a hyperbola. Similarly, in a chart $U_{x_2} = \{x_0 = 1\}$ with local affine coordinates $t_0 = x_0/x_2$, $t_1 = x_1/x_2$ we get the equation $t_0^2 + t_1^2 = 1$, i.e. $C \cap U_{x_2}$ is a circle. Finally, consider a chart $U_{x_2-x_1}$ given by $x_2 - x_1 = 1$ with local affine coordinates $t_0 = x_0|_{U_{x_2-x_1}} = \dfrac{x_0}{x_2 - x_1}$, $t_1 = (x_2 + x_1)|_{U_{x_2-x_1}} = \dfrac{x_2 + x_1}{x_2 - x_1}$. After dividing by $(x_2 - x_1)^2$ and some eliminations, we see that $C \cap U_{x_2-x_1}$ is the parabola $t_1 = t_0^2$.

1.4.2 Standard affine covering and gluing rules. Clearly, the whole of $\mathbb{P}_n$ is covered by $(n+1)$ affine charts $U_\nu \overset{\text{def}}{=} U_{x_\nu}$ given in $\mathbb{A}^{n+1}$ by equations $x_\nu = 1$. This covering is called *the standard affine covering*. For each $\nu = 0, 1, \ldots, n$ we take n forms

$$t_i^{(\nu)} = x_i|_{U_\nu} = \frac{x_i}{x_\nu}\,, \quad \text{where } 0 \leqslant i \leqslant n\,,\ i \neq \nu$$

as the standard local affine coordinates on U_ν. Topologically, this means that $\mathbb{P}_n$ is constructed from $(n+1)$ distinct copies of $\mathbb{A}^n$ denoted as $U_0, U_1, \ldots, U_n$ by gluing them together along the actual intersections $U_\mu \cap U_\nu \subset \mathbb{P}_n$ (i.e. a point of U_ν is identified with a point of U_μ under this gluing rules iff they correspond to the same point of $\mathbb{P}_n$). In term of the homogeneous coordinates, the intersection $U_\mu \cap U_\nu$ consists of all x such that both x_μ and x_ν are non zero. This locus is presented inside U_μ and U_ν by inequalities $t_\nu^{(\mu)} \neq 0$ and $t_\mu^{(\nu)} \neq 0$ respectively. Thus a point $t^{(\mu)} \in U_\mu$ is glued with a point $t^{(\nu)} \in U_\nu$ iff

$$t_\nu^{(\mu)} = 1/t_\mu^{(\nu)} \qquad \text{and} \qquad t_i^{(\mu)} = t_i^{(\nu)}/t_\mu^{(\nu)} \quad \text{for} \quad i \neq \mu, \nu\ .$$

The right hand sides of these equations are called *the transition functions* from $t^{(\nu)}$ to $t^{(\mu)}$ over $U_\mu \cap U_\nu$.

For example, $\mathbb{P}_1$ can be produced from two copies of $\mathbb{A}^1$ by identifying the point t in one of them with the point $1/t$ in the other for all $t \neq 0$.

EXERCISE 1.1*. If you have some experience in smooth topology, prove that real and complex projective lines are analytic manifolds isomorphic to the circle S^1 (in real case) and to the Riemann sphere S^2 (in complex case).

1.4.3 Example: projective closure. Any affine algebraic variety $X \subset \mathbb{A}^n$ is an affine piece $X = \widetilde{X} \cap U_{x_0}$ of projective algebraic variety $\widetilde{X} \subset \mathbb{P}_n$ viewed in standard affine chart U_{x_0} with local affine coordinates $t_i = x_i/x_0 = x_i|_{U_{x_0}}$. If X is given by a system of (non-homogeneous) polynomial equations

$$f_\nu(t_1, t_2, \ldots, t_n) = 0\,,$$

then $\widetilde{X} \subset \mathbb{P}_n$ is given by homogeneous equations $\widetilde{f}_\nu(x_0, x_1, \ldots, x_n) = 0$ of the same degrees $\deg \widetilde{f}_\nu = \deg f_\nu$ obtained from f_ν by the substitution $t_i = x_i/x_0$ and multiplication of the both sides in $f_\nu = 0$ by $x_0^{\deg f_\nu}$. Projective variety $\widetilde{X}$ is called *a projective closure of* X. Geometrically, $\widetilde{X}$ is the union of X with all its *asymptotic directions*. Thus, the projective langauge allows to treat affine asymptotic directions as ordinary points lying at infinity w.r.t. initial affine chart.

1.5 Projective subspaces. Projective algebraic variety given by a system of *linear* homogeneous equations is called *projective subspace*. In other words, a projective subspace $L \subset \mathbb{P}(V)$ is a projective space $L = \mathbb{P}(W)$ associated with some vector subspace $W \subset V$.

EXERCISE 1.2. Let $L \subset \mathbb{P}_n$ be k-dimensional projective subspace and $U_\xi \subset \mathbb{P}_n$ be an affine chart. Show that $L \cap U_\xi$ is either empty or k-dimensional affine subspace of U_ξ.

Since $\operatorname{codim}_{\mathbb{P}(V)} \mathbb{P}(W) = \operatorname{codim}_V W$, we have $L_1 \cap L_2 \neq \varnothing$ for any two projective subspaces L_1 and L_2 such that $\operatorname{codim} L_1 + \operatorname{codim} L_2 \leqslant n$. For example, any two lines on $\mathbb{P}_2$ have non empty intersection[1].

[1] in terms of $\mathbb{A}^3$ this means that any two planes containing the origin are intersected along a line

1.5.1 Complementary subspaces and projections. Two projective subspaces L_1 and L_2 in $\mathbb{P}_n$ are called *complementary* to each other, if $L_1 \cap L_2 = \varnothing$ and $\dim L_1 + \dim L_2 = n-1$. For example, any two skew lines in 3-dimensional space are complementary.

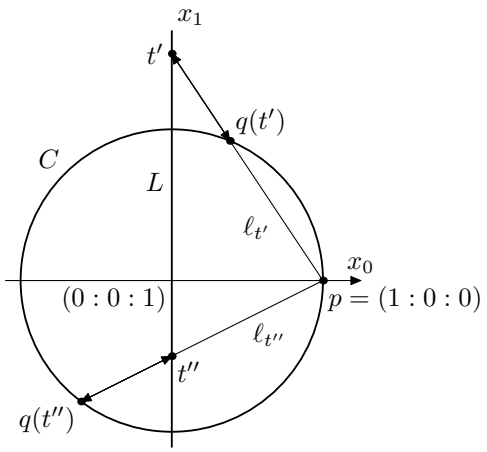

Fig. 1◇3. Projecting a conic.

Projective subspaces $K = \mathbb{P}(U)$ and $L = \mathbb{P}(W)$ are complementary in $\mathbb{P}(V)$ iff $V = U \oplus W$, because the above conditions say that $U \cap W = 0$ and $\dim U + \dim W = \dim V$. This means that any $v \in V$ has a unique decomposition $v = u + w$ with $u \in U$ and $w \in W$. If $v \notin U \cup W$, then u and w are non zero vectors and span a unique 2-dimensional subspace in V that contains v and has non zero intersections with U and with W. In geometric langauge this means that each point $p \in \mathbb{P}(V) \setminus (L_1 \cup L_2)$ lies on a unique line crossing both subspaces K and L.

Thus, there is well defined map $\pi_L^K : \mathbb{P}_n \smallsetminus K \longrightarrow\!\!\!\!\!\rightarrow L$ that identically acts on L and sends a point $p \in \mathbb{P}_n \setminus (L_1 \sqcup L_2)$ to $\ell \cap L_2$, where ℓ is the unique line passing through p and crossing K and L. It is called *projection from K onto L*.

In homogeneous coordinates $(x_0 : x_1 : \ \ldots\ : x_n)$ such that K is coordinated by $(x_0 : x_1 : \ \ldots\ : x_m)$ and L is coordinated by $(x_{m+1} : x_{m+2} : \ \ldots\ : x_n)$, projection π_L^K just removes the first $(m+1)$ coordinates in $(x_0 : x_1 : \ \ldots\ : x_n)$.

1.5.2 Example: projecting a conic onto a line. Consider the projection $\pi_\ell^p : Q \longrightarrow\!\!\!\!\!\rightarrow \ell$ of the plane conic (1-2) onto the line $\ell = \{x_0 = 0\}$ from the point $p = (1:0:1) \in Q$. Inside the standard affine chart U_2, where $x_2 = 1$, it looks like on fig. 1◇3. Being extended by the rule $p \mapsto \ell \cap T_pQ$, where T_pQ is the tangent line to Q at $p \in Q$, this projection provides a birational bijection between the conic and the line. Indeed, the pencil of all lines passing through p is bijectively parameterized by the points $t \in \ell$ and any such a line (pt) intersects Q exactly in one more point $q = q(t)$ in addition to p (except for the the tangent line at p, which is given by $x_0 = x_2$ and crosses ℓ at the point[1] $t = (0:1:0)$ corresponding to $q(t) = p$ itself). Coordinates of corresponding points $(q_0 : q_1 : q_2) \in Q$ and $(0 : t_1 : t_2) \in L$ are *rational* functions of each other:

$$(t_1 : t_2) = (\, q_1 : (q_2 - q_0)\,)$$
$$(q_0 : q_1 : q_2) = (\, (t_1^2 - t_2^2) : 2\,t_1 t_2 : (t_1^2 + t_2^2)\,)\,.$$

EXERCISE 1.3. Check these formulas and show that the second formula gives the full list of the Pythagorean triples[2] $(q_0 : q_1 : q_2)$ when (t_1, t_2) runs through $\mathbb{Z} \times \mathbb{Z}$.

1.6 Projective duality. Projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ are called *dual* and denoted by $\mathbb{P}_n$ and $\mathbb{P}_n^\times$ when a nature of V is not essential. Canonical isomorphism $V^{**} \simeq V$ leads to identification $\mathbb{P}_n^{\times\times} = \mathbb{P}_n$. Since the classes of proportional linear forms $\xi \in V^*$ correspond bijectively to vector subspaces $U = \mathrm{Ann}\,(\xi) \subset V$ of codimension 1, $\mathbb{P}_n^\times$ is nothing but *the space of hyperplanes* in $\mathbb{P}_n$ and vice versa. More generally, for each $m = 0, 1, \ldots, (n-1)$ there is a canonical bijection between m-dimensional subspaces in $\mathbb{P}_n$ and $(n-m-1)$-dimensional subspaces in $\mathbb{P}_n^\times$. It sends $L = \mathbb{P}(U)$ to $L^\times = \mathbb{P}(\mathrm{Ann}\,(U))$, where

$$\mathrm{Ann}\,(U) \stackrel{\text{def}}{=} \{\, \xi \in V^* \mid \xi(u) = 0 \quad \forall\, u \in U \,\}$$

is *annihilator* of U. Since $\mathrm{Ann}\,\mathrm{Ann}\,U = \{\, v \in V \mid \xi(v) = 0 \quad \forall\, \xi \in \mathrm{Ann}\,U \,\} = U$ under the natural identification $V^{**} \simeq V$, we have $L^{\times\times} = L$. Correspondence $L \leftrightarrow L^\times$ is called *projective duality*. It inverts inclusions[3] and linear incidences[4]. For example, for dual planes $\mathbb{P}_2$ and $\mathbb{P}_2^\times$ it leads to the

[1] this intersection point lies at the infinity on fig. 1◇3

[2] i.e. right triangles with integer sides

[3] i.e. $L_1 \subset L_2 \Longleftrightarrow L_1^\times \supset L_2^\times$

[4] i.e. $L_1, L_2, \ldots, L_r$ are contained in some m-dimensional subspace L iff $L_1^\times, L_2^\times, \ldots, L_r^\times$ contain some $(n-m-1)$-dimensional subspace $L^\times$; for example: 3 points are collinear iff their dual 3 hyperplanes have common subspace of codimension 2

following dictionary:

a line $\ell \subset \mathbb{P}_2$	$\longleftrightarrow$	a point $\ell^\times \in \mathbb{P}_2^\times$
the points p of the above line ℓ	$\longleftrightarrow$	the lines $p^\times$ passing through the above point $\ell^\times$
the line passing through two points $p_1, p_2 \in \mathbb{P}_2$	$\longleftrightarrow$	the intersection point for two lines $p_1^\times, p_2^\times \subset \mathbb{P}_2^\times$

1.7 Reminding matrix notations for linear maps. Let U and W be vector spaces over $\Bbbk$, $\dim U = n$, $\dim W = m$. Write $\mathrm{Hom}(U, W)$ for the space of $\Bbbk$-linear maps $U \longrightarrow W$ and $\mathrm{Mat}_{m\times n}(\Bbbk)$ for the space of matrices with m rows, n columns and elements from $\Bbbk$.

With any pair of bases $u_1, u_2, \ldots, u_n \in U$ and $w_1, w_2, \ldots, w_m \in W$ is associated isomorphism

$$\mathrm{Hom}(U, W) \xrightarrow{\sim} \mathrm{Mat}_{m\times n}(\Bbbk)$$

that takes linear operator $U \xrightarrow{F} W$ to its matrix $M_F = (a_{ij})$ whose j-th column consists of m coordinates of vector $F(u_j) \in W$ in basis $\{w_1, w_2, \ldots, w_m\}$. In other words, $F(u_j) = \sum\limits_{\nu=1}^{m} a_{\nu j}\, w_\nu$ or, using matrix multiplication, $(F(u_1),\ F(u_2),\ \ldots\ ,\ F(u_n)) = (w_1, w_2, \ldots, w_m) \cdot M_F$.

If $y = F(x)$, then ${}^t y = A \cdot {}^t x$, where ${}^t x$, ${}^t y$ are the coordinates of x and y arranged into columns. This follows at once from computation

$$\begin{aligned}(w_1, w_2, \ldots, w_m) \cdot {}^t y = F(x) = F\big((u_1, u_2, \ldots, u_n) \cdot {}^t x\big) = \\ = \big(F(u_1),\ F(u_2),\ \ldots\ ,\ F(u_n)\big) \cdot {}^t x = (w_1, w_2, \ldots, w_m) \cdot M_F \cdot {}^t x\,.\end{aligned}$$

1.8 Linear projective isomorphisms. Each linear isomorphism of vector spaces $U \xrightarrow{F} W$ produces bijective map $\mathbb{P}(U) \xrightarrow{\overline{F}} \mathbb{P}(W)$ called *projective linear isomorphism* provided by F. By the definition, it takes a point represented by vector $u \in \mathbb{P}(U)$ to the point represented by vector $F(u) \in W$.

EXERCISE 1.4. Let $\ell_1, \ell_2 \in \mathbb{P}_2$ be two lines and point $p \in \mathbb{P}_2$ do not belong them. Check that central projection $\gamma_p : \ell_1 \longrightarrow \ell_2$, which takes $t \in \ell_1$ to $(tp) \cap \ell_2$, is a projective linear isomorphism.

Let us say that points $p_1, p_2, \ldots, p_m \in \mathbb{P}_n = \mathbb{P}(V)$ are *linearly generic*, if any k of them span $(k-1)$ dimensional projective subspace for each k in range $2 \leqslant k \leqslant \min(m, n+1)$. Thus, $(n+1)$ points are linearly general iff the corresponding vectors form a basis in V.

LEMMA 1.2
For any two ordered collections of $(n+2)$ linearly generic points

$$p_0, p_1, \ldots, p_{n+1} \in \mathbb{P}(U) \quad \text{and} \quad q_0, q_1, \ldots, q_{n+1} \in \mathbb{P}(W)$$

in projective spaces $\mathbb{P}(U)$ and $\mathbb{P}(W)$ of the same dimension $n = \dim \mathbb{P}(U) = \dim \mathbb{P}(W)$ there exists a unique up to proportionality linear isomorphism $F : V \longrightarrow W$ such that associated projective linear isomorphism $\overline{F} : \mathbb{P}(U) \xrightarrow{\sim} \mathbb{P}(W)$ takes $p_i \longrightarrow q_i$.

PROOF. Fix some vectors u_i and w_i representing the points p_i and q_i and consider $u_0, u_1, \ldots, u_n$ and $w_0, w_1, \ldots, w_n$ as the bases in U and W. Since both collections of points are linearly generic, in the linear expansions of vectors u_{n+1} and w_{n+1} through these bases

$$\begin{aligned} u_{n+1} &= x_0 u_0 + x_1 u_1 + \cdots + x_n u_n \\ w_{n+1} &= y_0 w_0 + y_1 w_1 + \cdots + y_n w_n \end{aligned} \tag{1-3}$$

all coefficients x_i and y_i are non zero. Linear map $F \in \mathrm{Hom}(U, W)$ provides projective linear isomorphism that takes $F(p_i) = q_i$ iff $F(u_i) = \lambda_i w_i$ for some non zero $\lambda_i \in \Bbbk$. For $i = n+1$ this leads to

$n+1$ equations $\lambda_{n+1}x_i = \lambda_i y_i$ on the coordinates (1-3). Thus, $\lambda_i = \lambda_{n+1}x_i/y_i$ for each $0 \leqslant i \leqslant n$, i.e. the matrix of F in the bases u_i and w_i is a diagonal matrix of form

$$M_F = \lambda_{n+1} \cdot \begin{pmatrix} \frac{x_0}{y_0} & & & \\ & \frac{x_1}{y_1} & & \\ & & \ddots & \\ & & & \frac{x_0}{y_0} \end{pmatrix}$$

where $\lambda_{n+1} \in \Bbbk$ is an arbitrary non zero constant. □

Corollary 1.1
Two linear isomorphisms $F, G : U \xrightarrow{\sim} V$ produce the same projective linear isomorphism $\overline{F} = \overline{G}$ iff $F = \lambda b$ for some non zero constant $\lambda \in \Bbbk$.

1.8.1 Linear projective group. Linear automorphisms $V \xrightarrow{\sim} V$ form a group which is denoted by $\mathrm{GL}(V)$ and called *general linear group* of V. It acts on $\mathbb{P}(V)$ by the linear projective isomorphism and the kernel of this action coincides with the subgroup of scalar dilatations $H \subset \mathrm{GL}(V)$. Thus the group of all projective linear automorphisms of $\mathbb{P}(V)$ is the factor group $\mathrm{GL}(V)/H$. It is denoted by $\mathrm{PGL}(V)$ and called *projective linear group* of V.

Fixation of some basis $e_0, e_1, \ldots, e_n \in V$ identifies $\mathrm{GL}(V)$ with the group

$$\mathrm{GL}_{n+1}(\Bbbk) \subset \mathrm{Mat}_{n+1}(\Bbbk)$$

of non degenerated square matrices. Under this identification the dilatations go to the scalar diagonal matrices and $\mathrm{PGL}(V)$ turns into the group

$$\mathrm{PGL}_{n+1}(\Bbbk) \stackrel{\text{def}}{=} \mathrm{GL}_{n+1}(\Bbbk)/\{\text{scalar diagonal matrices } \lambda E\}$$

of classes of non degenerate square matrices up to proportionality.

1.9 Linear fractional group and cross ratio. Group $\mathrm{PGL}_2(\Bbbk)$ consists of all 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with } ad - bc \neq 0$$

considered up to proportionality. It acts on $\mathbb{P}_1$ via $(x_0 : x_1) \longmapsto \big((ax_0 + bx_1) : (cx_0 + dx_1)\big)$. In the standard affine chart U_{x_1} with affine coordinate $t = x_0/x_1$ this action looks like the linear fractional transformation $t \longmapsto (at + b)/(ct + d)$.

Exercise 1.5. Verify by the straightforward computation that $(AB)(t) = A(B(t))$.

It follows from lem. 1.2 that there exists a unique linear fractional transformation that sends an arbitrary triple of distinct points p, q, r to ∞, 0, 1 respectively. It takes

$$t \longmapsto \frac{t-r}{t-q} \cdot \frac{s-r}{s-q} \tag{1-4}$$

The right hand site of (1-4) is called *a cross ratio* of 4 points q, r, s, t on $\mathbb{P}_1$ and is denoted by $[q, r, s, t]$. In homogeneous coordinates the cross ratio can be expressed in terms of 2×2 determinants

$$[p_1, p_2, p_3, p_4] = \frac{(p_1 - p_3)\,(p_2 - p_4)}{(p_1 - p_4)\,(p_2 - p_3)} = \frac{\det(p_1, p_3) \cdot \det(p_2, p_4)}{\det(p_1, p_4) \cdot \det(p_2, p_3)}, \tag{1-5}$$

where we write $\det(a, b) = \det \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \end{pmatrix} = a_0 b_1 - a_1 b_0$ for two points $a = (a_0 : a_1)$, $b = (b_0 : b_1)$ on $\mathbb{P}_1$. It is clear from the definition that cross ratio of 4 distinct points can take any value except for ∞, 0 и 1 and a quadruple of points can be mapped to another quadruple of points by a linear fractional

automorphism of line iff these quadruples have equal cross ratios. Since a change of homogeneous coordinates is also a linear fractional automorphism, the cross ratio (1-5) does not depend on a choice of coordinates.

EXERCISE 1.6. Verify by a direct computation that the cross ratio is preserved under a linear change of homogeneous coordinates.

Let us ascertain a behavior of the cross ratio under permutations of points. It follows from (1-5) that the Klein subgroup $V_4 \subset S_4$, which consists of identity an 3 pairs of independent transpositions, does not change the cross ratio:

$$[p_1, p_2, p_3, p_4] = [p_2, p_1, p_4, p_3] = [p_3, p_4, p_2, p_1] = [p_4, p_3, p_2, p_1] \tag{1-6}$$

Hence the action of symmetric group S_4 on the value of cross ratio is factorized through surjection

$$S_4 \longrightarrow\!\!\!\!\!\rightarrow S_4/V_4 \simeq S_3\,.$$

The latter group is formed by the class of identity, classes of transpositions $(1,2)$, $(1,3)$, $(1,4)$, and classes of cycles (123) and (132). Writing ϑ for the value of cross ratio (1-6) , we get

$$\begin{aligned}
&[p_1, p_2, p_3, p_4] = [p_2, p_1, p_4, p_3] = [p_3, p_4, p_2, p_1] = [p_4, p_3, p_2, p_1] = \vartheta\\
&[p_2, p_1, p_3, p_4] = [p_1, p_2, p_4, p_3] = [p_3, p_4, p_1, p_2] = [p_4, p_3, p_1, p_2] = 1/\vartheta\\
&[p_3, p_2, p_1, p_4] = [p_2, p_3, p_4, p_1] = [p_1, p_4, p_2, p_3] = [p_4, p_1, p_2, p_3] = \vartheta/(\vartheta-1)\\
&[p_4, p_2, p_3, p_1] = [p_2, p_4, p_1, p_3] = [p_3, p_1, p_2, p_4] = [p_1, p_3, p_2, p_4] = 1-\vartheta\\
&[p_2, p_3, p_1, p_4] = [p_3, p_2, p_4, p_1] = [p_1, p_4, p_3, p_2] = [p_4, p_1, p_3, p_2] = (\vartheta-1)/\vartheta\\
&[p_3, p_1, p_2, p_4] = [p_1, p_3, p_4, p_2] = [p_2, p_4, p_1, p_3] = [p_4, p_2, p_1, p_3] = 1/(1-\vartheta)\,.
\end{aligned} \tag{1-7}$$

EXERCISE 1.7. Check all these equalities.

There are 3 special values $\vartheta = -1,\ 2,\ 1/2$ that are not changed under transpositions $(1,2)$, $(1,3)$ и $(1,4)$ and are cyclically permuted by cycles of length 3. Two other special values of ϑ are equal to the roots of $x^2 - x + 1 = 0$ (i.e. two cubic roots of unity except for -1). These two values are stabilized by cycles of length 3 and are exchanged by transpositions. All other values of ϑ produce 6 distinct values of cross ratio under the permutations of the points.

1.9.1 Harmonic pairs of points. Four points $a, b, c, d \in \mathbb{P}_1$ are called *harmonic*, if $[a, b, c, d] = -1$. In this case one says also that two pairs of points (a, b) and (c, d) are *harmonic* to each other. It follows from above computation that this is a symmetric relation on the set of *unordered* pairs of points on a projective line. Moreover, harmonicity of pairs (a, b) and (c, d) is equivalent to identity $[a, b, c, d] = [b, a, c, d]$.

Geometrically, harmonicity of pairs (a, b) and (c, d) means that in affine chart where $a = \infty$ point b is the middle point of segment $[c, d]$.

1.9.2 Example: complete quadrangle. Each quadruple of points $a, b, c, d \in \mathbb{P}_2$, in which any 3 points are non collinear, produces 3 pairs of lines joining the pairs of these points (see fig. 1⋄4). The lines in each pair are called *opposite sides* of quadrangle *abcd*. They are crossing at 3 more points $x = (ab) \cap (cd)$, $y = (ac) \cap (bd)$, $z = (ad) \cap (bc)$.

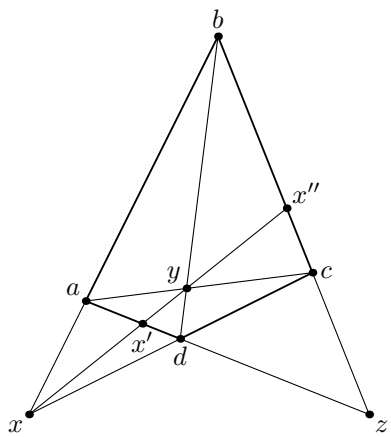

Fig. 1⋄4. Quadrangle.

Let us parameterize the pencil of lines passing through x by the points of line (ad). Since central projections from x and y produce linear projective isomorphisms between lines (ad) and (bc), we have the equalities $[a, d, z, x'] = [b, c, z, x''] = [d, a, z, x']$, which imply that $[a, d, z, x'] = [b, c, z, x''] = -1$.

In other words, the pair of opposite sides of quadrangle *abcd* is harmonic in the pencil of lines passing through x to the pair of sides of triangle xyz crossing at x. Of course, the same property holds at the points y and z as well.