Final Written Exam

Problems can be solved in any ordering. Complete solution of each problem contributes 25 points to the final result. Thus, to reach the 100% result, it's enough to solve completely any 4 of 5 problems listed below.

Problem 1. Fix 6 points $\{p_1, p_2, \dots, p_6\} \subset \mathbb{P}_2 = \mathbb{P}(V)$ such that any 3 of them are not collinear and all 6 do not lie on a conic. Write $\mathbb{P}(W)$ for the space of all cubic curves passing through $p_1, p_2, \dots, p_6$. Thus, $W = \{F \in S^3V^* \mid F(p_i) = 0 \text{ for each } i = 1,\, 2,\, \dots\, , 6\}$. For any point $p \in \mathbb{P}_2 \smallsetminus \{p_1, p_2, \dots, p_6\}$ all cubic curves $C \in \mathbb{P}(W)$ passing through p form a hyperplane in $\mathbb{P}(W)$. Algebraically, this hyperplane is the zero set of the linear form $\mathrm{ev}_p : F \longmapsto F(p)$ on W. Sending p to this hyperplane, we get a map $\psi : \mathbb{P}_2 \smallsetminus \{p_1, p_2, \dots, p_6\} \xrightarrow{\mathbb{P}} (W^*)$

a) Show that $\dim W = 4$

b) Show that the closure $\overline{(\mathrm{im}\,(\psi))}$ is a cubic surface S in $\mathbb{P}_3 = \mathbb{P}(W^*)$.

c) Find inside W 27 pencils of cubics dual to 27 lines on S

Problem 2. Let a finite group G act on an affine algebraic variety X by regular automorphisms.

a) Construct an affine algebraic variety X/G and a finite regular surjection $X \twoheadrightarrow X/G$ whose fibers are exactly G-orbits.

b) Show that X/G is universal in the following sense: for any regular morphism of affine algebraic varieties $X \xrightarrow{\varphi} Y$ such that $\varphi(gx) = \varphi(x)$ for all $g \in G$ and all $x \in X$ there exists a unique regular morphism $G/X \xrightarrow{\psi} Y$ such that $\psi\pi = \varphi$.

c) Let the symmetric group S_n act on the affine space $\mathbb{A}^n$ by the permutations of coordinates. Describe $\mathbb{A}^n/S_n$.

Problem 3. Let us fix $(n+1)$ degrees $d_0, d_1, \dots, d_n$ and write $\mathbb{P}_{N_i} = \mathbb{P}S^{d_i}V^*$ for the space of degree d_i hypersurfaces in $\mathbb{P}_n = \mathbb{P}(V)$. Show that

a) $\Gamma = \{(S_0, S_1, \dots, S_n,\, p) \in \mathbb{P}_{N_0} \times \cdots \mathbb{P}_{N_n} \times \mathbb{P}_n \mid p \in \bigcap_{\nu=0}^{n} S_\nu\}$ is an irreducible projective variety and find $\dim \Gamma$

b) there exists an irreducible polynomial R in the coefficients of homogeneous forms $F_0, F_1, \dots, F_n$ of degrees $d_0, d_1, \dots, d_n$ in variables $(x_0, x_1,\, \dots,\, x_n)$ such that $R = 0$ iff the system of equations $F_\nu(x_0, x_1,\, \dots,\, x_n) = 0$ (where $0 \leqslant \nu \leqslant n$) has a non zero solution. (By the way: how does R look like for a system of linear equations?)

Problem 4. Write M for the projective space of $m \times n$ matrices considered up to proportionality. Use appropriate incidence variety $\Gamma = \{(L, F) \mid L \subset \ker F\}$ (where $L \subset \Bbbk^n$ is a subspace and $F : \Bbbk^n \longrightarrow \Bbbk^m$ is linear operator represented by a matrix from M) to show that the matrices of rank $\leqslant k$ form an irreducible projective subvariety $M_k \subset M$ and to find $\dim M_k$.

Problem 5. Let us define the cotangent bundle to $\mathbb{P}_1$ as a bundle whose sections over a standard affine chart $\mathbb{A}^1$ with affine coordinate t are differential forms $f(t)dt$ with polynomial coefficients $f(t) \in \Bbbk[t]$. Similarly, define the tangent bundle to $\mathbb{P}_1$ as a bundle whose sections over the same chart are differential operators $f(t)\frac{d}{dt} : \Bbbk[t] \longrightarrow \Bbbk[t]$ with polynomial coefficients $f(t) \in \Bbbk[t]$. Identify the both tangent and cotangent bundles with appropriate line bundles $\mathscr{O}(d)$.